Nature of the Bañados – Silk – West effect

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Kinematic explanations
Geometric explanation

Kinematics of the BSW effect

When two particle collide near BH, their energy in CM can grow unbound. Why ?

Metric $$ds^2 = -N^2 dt^2 + g_{\phi\phi}(d\phi - \omega dt)^2 + dl^2 + g_{zz} dz^2.$$ Horizon N=0

Metric coefficients do not depend on t and ϕ

$\theta = \frac{\pi}{2}$ equatorial plane

Equations of motion, m=1

$$\dot{t} = u^0 = \frac{X}{N^2}, \qquad X = E - \omega L$$

$$\dot{\phi} = \frac{L}{g_{\phi\phi}} + \frac{\omega X}{N^2}, \qquad \dot{l}^2 = \frac{(E - \omega L)^2}{N^2} - 1 - \frac{L^2}{g_{\phi\phi}}.$$

Forward in time condition $\dot{t} > 0$ $\dot{t} = \frac{X}{N^2}$, $X = E - \omega L$

Classification of particles

Usual $X_H > 0$

Critical $X_H = 0$

Near-critical $X_H > 0$ but is small

Choice of tetrad basis

$x^0 = t, x^1 = l$ $x^2 = z$ $x^3 = \phi$

$$g_{\mu\nu} = h^{(a)}{}_{\mu} h_{(a)\nu}$$

Zero angular momentum observer (ZAMO)

Equivalent name: locally nonrotating frame (LNRF)

$$h_{(0)\mu} = -N(1,0,0,0), \qquad h_{(1)\mu} = (0,1,0,0), \qquad x^3 = \phi$$

$$h_{(2)\mu} = \sqrt{g_{zz}}(0,0,1,0), \qquad h_{(3)\mu} = \sqrt{g_{\phi\phi}}(-\omega,0,0,1)$$

$$h_{(0)}{}^{\mu} = \frac{1}{N}(1,0,0,\omega), \qquad h_{(1)}{}^{\mu} = (0,1,0,0),$$

$$h_{(2)}{}^{\mu} = \frac{1}{\sqrt{g_{zz}}}(0,0,1,0), \qquad h_{(3)}{}^{\mu} = \frac{1}{\sqrt{g_{\phi\phi}}}(0,0,0,1)$$

ZAMO $U^{\mu} = h_{(0)}{}^{\mu} = \dfrac{1}{N}(1,0,0,\omega),$ $U_{\mu} = -N(1,0,0,0),$

$$\frac{d\phi}{dt} = \frac{U^{\phi}}{U^{t}} = \omega \quad \neq 0 \qquad U_{\phi} = 0$$

Rotates with geometry

$$ds^2 = -N^2 dt^2 + g_{\phi\phi}(d\phi - \omega dt)^2 + dl^2 + g_{zz} dz^2.$$

Three-velocity

$$v^{(i)} = v_{(i)} = \frac{u^{\mu} h_{\mu(i)}}{-u^{\mu} h_{\mu(0)}} = \frac{dx^{\mu} h_{\mu(i)}}{d\tau_{obs.}}.$$

$$d\tau_{obs.} = -dx^{\mu} h_{\mu(0)} = N dt$$

$$-u_\mu h^\mu_{(0)} = \frac{E-\omega L}{N}, \qquad u_\mu h^\mu_{(3)} = \frac{L}{\sqrt{g_{\phi\phi}}}.$$

$$v^{(1)} = \sqrt{1-\frac{N^2}{(E-\omega L)^2}(1+\frac{L^2}{g_{\phi\phi}})}. \qquad v^{(3)} = \frac{LN}{\sqrt{g_{\phi\phi}}\,(E-\omega L)},$$

$$v^2 = \left[v^{(1)}\right]^2 + \left[v^{(2)}\right]^2$$

$$E-\omega L = \frac{mN}{\sqrt{1-v^2}}, \qquad v^2 = 1-\left(\frac{mN}{E-\omega L}\right)^2.$$

Limiting transitions for relative velocity

$$E_{c.m.}^2 = -(p_1^\mu + p_2^\mu)(p_{1\mu} + p_{2\mu}) = m_1^2 + m_2^2 - 2m_1 m_2 u_1^\mu u_{2\mu}.$$

CM centre of mass

$$\gamma = -u_1^\mu u_{2\mu} = \frac{1}{\sqrt{1 - w^2}}$$

w relative velocity

Unbounded energy in CM frame: $w \to 1$ $\gamma \to \infty$

Some formulas of special relativity

$$\vec{v}_1 = v_1 \vec{n}_1 \qquad \vec{v}_2 = v_2 \vec{n}_2 \qquad w^2 = 1 - \frac{(1 - v_1^2)(1 - v_2^2)}{[1 - v_1 v_2 (\vec{n}_1 \vec{n}_2)]^2}$$

a) $v_1 \to 1,$ $v_2 < 1$ $(\vec{n}_1\vec{n}_2)$ is arbitrary $w \to 1$

b) $v_1 \to 1$ $v_2 \to 1$ in such a way that $v_i = 1 - A_i\delta$ $\delta \ll 1$

A_i constants

b1) $(\vec{n}_1\vec{n}_2) \neq 1$ $w^2 \approx 1 - \frac{4A_1A_2\delta^2}{[1-(\vec{n}_1\vec{n}_2)]^2},$ $w \to 1$

b2) $(\vec{n}_1\vec{n}_2) = 1$ $w \approx \frac{|A_1 - A_2|}{A_1 + A_2} < 1.$ $w < 1$

c) $v_1 < 1$ $v_2 < 1$ $(\vec{n}_1\vec{n}_2)$ is arbitrary $w < 1$

Asymptotics near horizon of extremal black hole

Usual particle $E - \omega_H L \neq 0$

$$E - \omega L = \frac{mN}{\sqrt{1 - v^2}}, \qquad N \to 0 \qquad v \to 1$$

$$v^{(3)} = \frac{LN}{\sqrt{g_{\phi\phi}}(E - \omega L)} \to 0 \qquad \vec{n}$$ pointed in “radial” direction

For any two such particles $(\vec{n}_1 \vec{n}_2) = 1.$ Collision of 2 usual particles – no BSW effect

Critical particle

$$\omega = \omega_H - B_1 N + B_2 N^2 + \ldots \qquad N \sim r - r_H$$

Critical particle $E - \omega_H L = 0$ $E - \omega L = \frac{mN}{\sqrt{1 - v^2}},$

$v^2 = 1 - \frac{1}{L^2 B_1^2} < 1$

$v^{(1)}$ and $v^{(3)}$ have the same order

Collision between two usual particles

case b2). $(\vec{n}_1 \vec{n}_2) = 1$ $w < 1$ No BSW effect

Collision between two critical particles

case c). $v_1 < 1$ $v_2 < 1$ $w < 1$ No BSW effect

Collision between usual and critical particles

Case a) $\boxed{w \to 1}$ $\gamma \to \infty$ BSW effect

$v_1 < 1$ Proper distance infinite, proper time is infinite

$$dl^2 \sim \frac{dr^2}{(r - r_H)^2} \qquad l \sim -\ln(r_H - r)$$

Rapid particle hits slow target

Collision between massive and massless particles

Previous explanation cannot be used directly since

since (i) there is no comoving frame for a massless particle

(ii) in any frame, such a particle moves always with the velocity of light.

Some modifications needed

Collision between massive particle (electron) and massless one (photon)

Motion of photons in equatorial plane $\theta = \frac{\pi}{2}$

$$\frac{dt}{d\lambda} = k^0 = \frac{\nu_0 - \omega L}{N^2}, \qquad \frac{d\phi}{d\lambda} = \frac{L}{g_{\phi\phi}} + \frac{\omega(\nu_0 - \omega L)}{N^2},$$

Affine parameter λ

$$\left(\frac{dl}{d\lambda}\right)^2 = \frac{(\nu_0 - \omega L)^2}{N^2} - \frac{L^2}{g_{\phi\phi}},$$

$\nu_0 = -k_0$ $\quad L = k_\phi$ $\quad k^\mu$ wave vector

ν_0 Constant. Frequency measured at inifnity

Forward – in - time condition $\nu_0 - \omega L > 0$

Energy in CM frame

$$E_{c.m.}^2 = -(p^\mu + k^\mu)^2$$

Planck constant ħ=1

$$k_\mu k^\mu = 0$$

$$E_{c.m.}^2 = m^2 - 2m(uk),\ (uk) \equiv u^\mu k_\mu.$$

$$-(uk) = \frac{X_1 X_2 - Z_1 Z_2}{N^2} - \frac{L_1 L_2}{g_{\phi\phi}},$$

1 is electron, 2 is photon

$$X_1 \equiv E_1 - \omega L_1 \qquad X_2 = \nu_0 - \omega L_2$$

$$Z_i = \sqrt{X_i^2 - N^2 b_i},\ b_1 = 1 + \frac{L_i^2}{g_{\phi\phi}},\ b_2 = \frac{L_2^2}{g_{\phi\phi}},$$

$$k^{(i)} = k_{(i)} = k^\mu h_{\mu(i)}, k^{(0)} = k^\mu h_\mu^{(0)} = -k^\mu h_{\mu(0)}.$$ light-like vector

$$k^{(1)} = -\sqrt{v^2 - \frac{L^2}{g_{\phi\phi}}}$$ $$k^{(3)} = \frac{L}{\sqrt{g_{\phi\phi}}},$$ ZAMO

$$v = \frac{v_0 - \omega N}{N}.$$ $$E - \omega L = \frac{mN}{\sqrt{1 - v^2}},$$ Massles versus massive

$$k^2 = \left[k^{(1)}\right]^2 + \left[k^{(2)}\right]^2$$ $$k^{(0)} = -k_\mu h^\mu_{(0)} = \frac{v_0 - \omega L}{N} = v$$

$$k^2 = \frac{(v_0 - \omega L)^2}{N^2} = v^2$$ $$k^2 - \left(k^{(0)}\right)^2 = 0$$

$$\vec{n}_2 = \frac{\vec{k}}{k}$$ Horizon limit: $(\vec{n}_1 \vec{n}_2) = 1$ when both usual

$(\vec{n}_1 \vec{n}_2) \neq 1$ otherwise

Different types of collisions

Case 1: electron is critical, photon is usual

Let us pass to the frame which is comoving with respect to the electron. Then, the frequency ν' measured in this frame is related to the frequency

in the ZAMO frame by the standard relativistic formula

$$\nu' = \gamma(\nu - \vec{k}\vec{v}) = \nu\gamma[1 - V(\vec{n}_1\vec{n}_2)].$$

For a critical particle $v \neq 1$ Lorentz factor γ is finite

$(\vec{n}_1\vec{n}_2) \neq 1$ Photon usual $\nu \rightarrow \infty$ when $N \rightarrow 0$

$$\nu = \frac{\nu_0 - \omega L}{N}. \qquad \nu' \rightarrow \infty$$

BSW effect

Result of two factors.

Infinite blue shift of radiation due to strong gravitating field near a black hole

red shift due to the Doppler effect since in the laboratory frame a receiver of radiation is moving apart from a photon

$$v^{(1)} < 0 \qquad k^{(1)} < 0$$

first factor is infinite second one is finite

net outcome is due to blue shift.

Case 2: electron is usual, photon is critical

$$\nu = \frac{\nu_0 - \omega L}{N}. \qquad \nu_0 - \omega_H L = 0 \qquad \nu \qquad \text{finite}$$

$$\nu' = \gamma(\nu - \vec{k}\vec{v}) = \nu\gamma[1 - V(\vec{n}_1\vec{n}_2)].$$

Electron is usual $V \to 1$ $\gamma \to \infty$

$(\vec{n}_1\vec{n}_2) \neq 1$ $\nu' \to \infty$

Let in a flat space-time a photon with the frequency ν

propagate in the laboratory frame and some observer moves with the velocity V with respect to it Then, in in its own frame, the observer measures the frequency of the process which is equal to ν' $(\vec{n}_1\vec{n}_2) \neq 1$

For simplicity, we can take $(\vec{n}_1\vec{n}_2) = 0$

Then, the frequency measured in the frame of a receiver

$$\nu' = \nu\gamma > \nu$$ due to the transverse Doppler effect

In the limit $V \to 1$ the Lorentz factor $\gamma \to \infty$

$$\nu' \to \infty$$

Even despite a moderate gravitational blue shift that resulted in a finite

ν the net outcome is infinite due to the Doppler effect.

BSW effect

Case 3: both particles are critical

$V<1$ $\quad \nu$ finite $\quad \nu' = \gamma(\nu - \vec{k}\vec{v}) = \nu\gamma[1 - V(\vec{n}_1\vec{n}_2)].$

ν' also finite

No BSW effect

Both factors - gravitational blue shifting and the Doppler effect are restricted and cannot give rises to infinite energies

Case 4: both particles are usual

$$E-\omega L=mN\gamma,\qquad \gamma=\frac{1}{\sqrt{1-v^2}}\qquad \nu=\frac{\nu_0-\omega L}{N}.$$

$$\nu\sim\frac{1}{N}\qquad \gamma\sim\frac{1}{N}$$

$$k^{(1)}=-\sqrt{\nu^2-\frac{L^2}{g_{\phi\phi}}}\qquad k^{(3)}=\frac{L}{\sqrt{g_{\phi\phi}}},$$

$$v^{(1)}=\sqrt{1-\frac{N^2}{(E-\omega L)^2}(1+\frac{L^2}{g_{\phi\phi}})}.\qquad v^{(3)}=\frac{LN}{\sqrt{g_{\phi\phi}}(E-\omega L)},$$

$$1-(\vec{n}_1\vec{n}_2)\sim N^2.\qquad \nu'=\gamma(\nu-\vec{k}\vec{v})=\nu\gamma[1-V(\vec{n}_1\vec{n}_2)].$$

Numerator and denominator compensate each other, ν' finite, no BSW effect

Play of two factors: Doppler effect (DE) and gravitational blue shift (GB)

Case 1: electron is critical, photon is usual	Finite redshift - DE . GB infinite Effect exists
Case 2: electron is usual, photon is critical	DE infinite, GB finite Effect exists
Case 3: both particles are critical	DE and GB finite, no effect
Case 4: both particles are usual	Infinite DE compensated by infinite GB Outcome is finite, no effect

Two viewpoints

1) ZAMO (static in limit of vanishing rotation)

2) Observer who falls in BH

How observer 2 explains BSW effect? Horizon is essential ingredient of BSW. But it does not reveal itself for observer 2.
Energy in CM frame is scalar. It cannot be very large in one frame and modest in another one.

More close examination of vicinity of horizon, relationship between two kinds of observers

Ideal tool – acceleration horizon, exact formulas

Appear on its own right in BH thermodynamics in limiting transition form nonextremal state to extremal, from rapidly rotating discs to black holes, AdS/CFT correspondence, etc.

Bertotti-Robinson space-time

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad f = \left(1 - \frac{r_+}{r}\right)^2$$

Extremal RN black hole

$$r = r_+ + \lambda x,\ t \to \frac{t}{\lambda} \qquad \text{Take limit} \qquad \lambda \to 0$$

$$ds^2 = -dt^2 \frac{x^2}{r_+^2} + r_+^2 \frac{dx^2}{x^2} + r_+^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Exact solution of Einstein-Maxwell equations

Horizon is pure kinematic effect

$$x = \sqrt{1+y^2}\cos\tilde{t} + y, \qquad t = \frac{\sqrt{1+y^2}\,\sin\tilde{t}}{\sqrt{1+y^2}\,\cos\tilde{t}+y}.$$

$$y = \frac{1}{2}\left(x + xt^2 - \frac{1}{x}\right), \qquad \cos\tilde{t} = \frac{1}{2}\frac{\left(x+\frac{1}{x}-xt^2\right)}{\sqrt{1+\frac{1}{4}\left(xt^2+x-\frac{1}{x}\right)^2}}.$$

$$ds^2 = -d\tilde{t}^2(1+y^2) + \frac{dy^2}{1+y^2} + d\theta^2 + \sin^2\theta d\phi^2.$$

BSW effect: general formulas

$$E_{c.m.}^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma \qquad\qquad \gamma = -u_1^{\mu} u_{2\mu},$$

$$\gamma = \frac{X_1 X_2 - Z_1 Z_2}{m_1 m_2 f}.$$

Near horizon f is small. Particle 1 is (near)critical,

$$\gamma \approx \frac{X_2(r_+)}{m_1 m_2 \sqrt{f}} \left(E_1 - \sqrt{E_1^2 - m_1^2}\right) \qquad\qquad f \sim x_c^2$$

$$\gamma \sim \frac{1}{x_c}.$$

Paradox of two frames

$$m\frac{d\tilde{t}}{d\tau} = \frac{\tilde{X}}{1+y^2}, \qquad m\frac{dy}{d\tau} = \tilde{Z}, \qquad \tilde{X} = \tilde{E} - q + qy,$$

$$\tilde{Z} = \sqrt{\tilde{X}^2 - m^2(1+y^2)}\,.$$

$$\gamma = \frac{\tilde{X}_1\tilde{X}_2 - \tilde{Z}_1\tilde{Z}_2}{1+y^2}, \qquad m_1 = m_2 = 1$$

. For any fixed energies $\tilde{E}_{1,2}$ and for any y

Properties of critical and usual paricles separately (definition from original frame)

Critical particle

$$X_1(r_+) = 0 \qquad E_1 = q_1 \equiv q \qquad X_1 = qx$$

$$x_1 = x_0 \exp(-\lambda\tau), \lambda = \sqrt{q^2 - 1}, \qquad t_1 = \frac{q\exp(\lambda\tau)}{x_0\lambda} = \frac{q}{\lambda x_1},$$

$$\tilde{X}_1 = E_1 y_1, \tilde{E}_1 = E_1 = q. \qquad y_1 = \frac{1}{2}(\frac{1}{x_1}\frac{1}{\lambda^2} + x_1).$$

Usual particle

$$x_2 = -E_2 \sin\tau, \qquad t_2 = -\frac{1}{E_2}\cot\tau + t_0 = \frac{1}{x}\sqrt{1-\frac{x^2}{E_2^2}} + t_0.$$

$\tau = 0$ at the moment of crossing the horizon

$\tau < 0$ before it.

$$y_2 = t_0\cos\tau - p\sin\tau = \frac{p}{E_2}x + t_0\sqrt{1-\frac{x^2}{E_2^2}},$$

$$p = \frac{1}{2}[(E_2 - \frac{1}{E_2}) + E_2 t_0^2].$$

$$\tilde{X}_2 = \tilde{E}_2 = \frac{1}{2}(E_2 + \frac{1}{E_2} + t_0^2 E_2).$$

usual does not coincide

$$\tilde{X}_1 = E_1 y_1, \tilde{E}_1 = E_1 = q.$$

critical coincides in both frames

Explanation of the paradox

(i) both particles should meet in the same point and
(ii) this point should be near the horizon.

large constant of integration t_0 (see below)

and the very large energy $\tilde{E}_2 \sim t_0^2$

$$t_1 = t_2, x_1 = x_2, \qquad \tilde{t}_1 = \tilde{t}_2, y_1 = y_2.$$

Collision at small $x = x_c$ $\qquad t_0 = \dfrac{E - \lambda}{x_c \lambda} + O(x_c)$

$$t_0 \sim x_c^{-1} \to \infty. \qquad p \sim \tilde{X}_2 \sim t_0^2 \sim \frac{1}{x_c^2} \to \infty$$

Second frame

$y = y_c$ $\qquad y_c \sim \frac{1}{x_c} \to \infty.$ $\qquad 1 \ll y_c \ll \tilde{X}_2.$

grows much faster than y,

$\tilde{X}_2 \sim t_0^2 \sim x_c^{-2}$ $\qquad \gamma \approx \frac{E_1 - \sqrt{E_1^2 - 1}}{y_c} \tilde{X}_2 \sim x_c^{-1}$

X_2 is not invariant under transformation Finite in frame 1, unbound in frame 2

Two alternative descriptions

Frame 1 horizon, finite energies

Frame 2 no horizons, unbound energy of usual particle

Energy conservation

$$X = E - q\varphi = \frac{N}{\sqrt{1 - V^2}}. \qquad V \to 1 \qquad \text{usual}$$

$$V \neq 1 \qquad \text{critical}$$

Frame 2

$$E_1 y = \frac{\sqrt{1 + y^2}}{\sqrt{1 - V_1^2}}.$$

As one approaches horizon, $y \to \infty$

$$V_1 = \sqrt{1 - \frac{1}{E_1^2}} < 1,$$

Usual particle. $V \to 1$

Relative velocty approaches 1

Simplified example: Minkowski and Rindler metrics

$$ds^2 = -dt^2 x^2 + dx^2,$$

Consider quadrant $t \geq 0 \quad x \geq 0$

$$ds^2 = -d\tilde{t}^2 + dy^2.$$

$$y = x \cosh t,$$

$$\tilde{t} = x \sinh t$$

$$x^2 = y^2 - \tilde{t}^2, \qquad \tanh t = \frac{\tilde{t}}{y}.$$

Horizon $x = 0$ corresponds to $y = \pm\tilde{t}$

Killing vector $\tilde{\xi}^\mu = (1,0)$ In Minkowskian coordinates

Another Killing vector $\xi^\mu = (1,0)$ in the Rindler coordinates

In Minkowskian it reads $\xi^\mu = (y, \tilde{t}).$

Geodesics $y = V\tilde{t} + y_0, \qquad t = \tau\tilde{E}$

$$\tilde{E} = \frac{1}{\sqrt{1-V^2}} \qquad \text{m=1}$$

$$E = -u_\mu \xi^\mu \quad \text{and} \quad \tilde{E} = -u_\mu \tilde{\xi}^\mu$$

$$E = \tilde{E} y_0.$$

$$x(\cosh t - V \sinh t) = y_0 \equiv \frac{E}{\tilde{E}}.$$

$$y = \frac{E\tilde{E}(\alpha + V\sqrt{\alpha^2 - 1})}{\alpha}, \ \alpha = \frac{E}{x}.$$

Collision

Rindler frame

$$\gamma = \frac{X_1 X_2 - Z_1 Z_2}{x^2},$$

$$Z_i = \sqrt{E_i^2 - x^2}$$

$X_i = E_i \neq 0$ no strictly critical particles

near-critical small X_1

Collision at small x_0 near horizon

If $E_1 = X_1 = O(x_0)$ $E_2 = O(1)$ $\gamma \sim x_0^{-1}$ grows unbound

Minkowski frame

$$\gamma = \tilde{E}_1 \tilde{E}_2 - \sqrt{\tilde{E}_1^2 - 1}\sqrt{\tilde{E}_2^2 - 1}.$$

is no small denominator here

Let, for simplicity, particle 2 have $V_2 = 0$

$y_2 = E_2 = const$ $\tilde{E}_2 = 1$ $\tilde{E} = \dfrac{1}{\sqrt{1 - V^2}}$

Let $E_1 \sim x_0 \ll 1$ so $\alpha \sim 1.$

Then, $\tilde{E}_1 \approx \dfrac{\tilde{E}_2}{E_1} \dfrac{\alpha}{\alpha + \sqrt{\alpha^2 - 1}} \sim x_0^{-1}$ grows unbound

BSW effect

For near-critical $V_1 \to 1$ in contrast to previous case

Rindler metric versus BR metric

Rapid particle hits slow one, horizon, finite individual energy

Effectively, no horizon, large individual energy of one of two paticles