УДК 530.21, 530.122

ДИНАМО-ОПТИЧЕСКИ АКТИВНЫЕ СРЕДЫ: НОВЫЕ АСПЕКТЫ ПОЛЕМИКИ МИНКОВСКОГО-АБРАГАМА

Альпин Т. Ю.[a,1], Балакин А. Б.[a,2]

[a] Кафедра теории относительности и гравитации, Институт физики, ФГАОУ ВО Казанский (Приволжский) федеральный университет, Казань, 420008, Россия

Две версии симметричного эффективного тензора энергии-импульса электромагнитного поля в динамо-оптически активной релятивистской среде построены в рамках тетрадной и эфирной парадигмы на основе ковариантного вариационного формализма. Показано, что скаляр плотности энергии и тензор давления совпадают для обеих версий тензора энергии-импульса, однако, векторы потока энергии оказываются, в общем случае, несовпадающими. Этот математический факт добавляет новую аргументацию в дискуссию, которая стартовала 100 лет назад, получила название полемика Минковского-Абрагама и связана с корректным определением потока электромагнитной энергии в сплошной среде. Мы рассматриваем три примера: во-первых, аксионно-активный вакуум, во-вторых, пространственно изотропная движущаяся диэлектрическая среда, в-третьих, динамо-оптически-активная среда. Упомянуты возможные приложения разработанного формализма.

Ключевые слова: анизотропная среда, эфирное расширение теории гравитации Эйнштейна, расширенные основополагающие уравнения.

DYNAMO-OPTICALLY ACTIVE MEDIA: NEW ASPECTS OF THE MINKOWSKI-ABRAHAM CONTROVERSY

Alpin T. Y.[a,1], Balakin A. B.[a,2]

[a] Department of General Relativity and Gravitation, Institute of Physics, Kazan (Volga region) Federal University, Kazan, Republic of Tatarstan, 420008, Russia

Based on the covariant variation formalism, two versions of the symmetric effective stress-energy tensor of the electromagnetic field in a dynamo-optically active relativistic media are reconstructed in the framework of the tetrad and aether paradigms, respectively. We show that the energy density scalars and pressure tensors coincide for both versions of the stress-energy tensors, however, the corresponding energy flux four-vectors happen to be different in general case. This mathematical fact adds new arguments into the 100-year-long discussion, which is called Minkowski-Abraham controversy and is connected with the correct definition of the electromagnetic energy flux in a continuous media. We consider three examples: first, the axionically active vacuum; second, the spatially isotropic moving dielectric medium; third, the dynamo-optically active medium. We discuss possible applications of the elaborated formalism.

Keywords: anisotropic medium, Einstein-aether theory, extended constitutive equations.

PACS: 04.40.-b , 04.40.Nr , 04.20.Jb , 98.80.Jk
DOI: 10.17238/issn2226-8812.2018.4.32-47

[1]E-mail: Timur.Alpin@kpfu.ru
[2]E-mail: Alexander.Balakin@kpfu.ru

Introduction

More than a century ago the term *Minkowski-Abraham controversy* appeared in the scientific lexicon as the result of discussions of Minkowski [1], Einstein and Laub [2], and Abraham [3]. These discussions were focused on the correct definition of the energy flux of the electromagnetic field in continuous material media. The interest to this problem was revived in 1950s - 1970s, in course of systematic elaboration of covariant theory of electromagnetically active media (see, e.g., [4]- [13]). In the review [14] Brevik formulated experimental motivation of the interest to this problem, thus giving a new impetus to investigations of the problem of electromagnetic energy transfer (see, e.g., [15]- [27] for the extension of discussions).

We attract attention of Readers to the problem of energy transfer in a Cosmic Dark Fluid, which joins the Dark Energy and Dark Matter constituents and can be considered as an electromagnetically active chiral medium [28]- [33]. The Dark Fluid is assumed to be electrically neutral, it does not contain electrically charged particles, however, this cosmic substratum, being a specific quasi-medium, can influence the electromagnetic field indirectly, and, respectively, can contribute its own electrodynamic part into the total stress-energy tensor of the Universe. One of the ways, which is open for the Dark Fluid influence, is the so-called dynamo-optical activity of the moving medium. This term was introduced in [42] to describe polarization and magnetization of a medium, which moves non-uniformly, i.e., when the medium flow is characterized by the acceleration, shear, rotation and expansion. When we deal with dynamo-optical interactions, we are faced with the problem how to separate the dynamo-optical energy flow and the one of the non-electromagnetic origin; in other words, we are faced again with the classical alternative associated with the Minkowski-Abraham controversy. There are at least three motives for studying the mentioned problem just now and namely in this context.

The first motif is connected with the definition of the velocity four-vector, which is the important player in the theory of the medium motion. On the one hand, there is the classical Landau-Lifshitz algebraic definition of the velocity four - vector V^i, appeared as the time-like eigen-vector of the medium stress-energy tensor; every cosmic constituent possesses such intrinsic velocity. On the other hand, as an alternative, there exists a global unit time-like vector field U^i, appeared in the Einstein-aether theory [34]- [36], which is associated with the velocity four-vector of some quasi-medium, the dynamic aether. This global vector field defines the preferred frame of reference [37–39], thus providing the violation of the Lorentz invariance of the theory [40, 41]. The model of dynamic aether is one of the candidates for describing the Dark Energy phenomenon [43].

The second motif relates to the axionic extension of the cosmic electrodynamics, which is associated with chirality of the cosmic medium. The pseudoscalar (axion) field interacts with the electromagnetic field, with vector field presenting the dynamic aether, and with gravitational field. When we study the waves in the cosmic medium, we deal, in fact, not simply with pure electromagnetic waves, but with a conglomerate of spin-0, spin-1 and spin-2 modes [33, 35]. The corresponding cross-terms in the total stress-energy tensor admit double interpretation, and we have to postulate: do they belong to the electromagnetic part of the stress-energy tensor, or, e.g., to the part associated with the axionic Dark Matter?

The third aspect is connected with the correct reconstruction of the stress-energy tensor of the electromagnetic field. There exist the canonic and effective stress-energy tensors of the system. The gravity field equations operate with the symmetric effective stress-energy tensor, which can be introduced using the variation procedure with respect to the space-time metric. Since, independently of definition, the velocity four-vector is considered to be normalized by unity, i.e., $g_{ik}V^iV^k = 1$, or $g_{ik}U^iU^k = 1$, this vector quantity depends on metric and thus has to participate in the variational procedure. Nevertheless, the variational procedures differ in the first and second cases; in order to distinguish them we use later two terms: the *tetrad paradigm*, and the *aether paradigm*, respectively. The first term reflects the fact that when the velocity is the eigen-vector of the stress-energy tensor, we can take it as the time-like unit vector $V^i = X^i_{(0)}$ of the corresponding tetrad $\left\{X^i_{(a)}\right\}$. The term aether paradigm relates to the case, when the velocity four-vector is associated with the unit time-like global vector field. In this context two questions arise. The first question is: whether the whole effective stress-energy tensors obtained by the variation procedure in the frameworks of the tetrad and aether paradigms, coincide? The second question is typical for the Minkowski-Abraham controversy: whether the electromagnetic energy flux vectors in the medium, obtained in the tetrad and aether paradigms, coincide? Why the corresponding difference can exist?

Also, we have to mention the following detail of discussion. The energy flux four-vector is known to appear as the result of application of the first or second projection procedure to the stress-energy tensor of the electromagnetic field (in the first procedure we project all the tensor quantities on the direction V^i and on hyper-surface orthogonal to it; in the second procedure we use the four-vector U^i). However, in the tetrad paradigm the V^i four-vector can be obtained as the eigen - vector either of the total stress-energy tensor, or, e.g., as the one for its pure material constituent, or for the Dark Fluid constituent. In other words, there exist an additional degree of freedom for modeling of this four-vector. In the aether paradigm the unique preferred global velocity four-vector plays this principal role, and there is no additional variants for the choice.

To conclude, there is no *a priori* fixed answer for the question concerning the structure and properties of the electromagnetic energy flux four-vector. The goal of this work is to clarify the posed questions using the model of the so-called dynamo-optical interactions in the framework of the Einstein-Maxwell-aether-axion theory.

The paper is organized as follows. In Section II we recall the schemes of derivation of the effective electromagnetic stress-energy tensors in the framework of the tetrad and aether paradigms. In Section III we derive the corresponding stress-energy tensors for the dynamo-optical interactions in the chiral electrodynamic systems. Section IV contains the analysis of the following three examples: the model of axionic vacuum, the model of spatially isotropic homogeneous moving dielectric medium, and the model of dynamo-optically active medium. We discuss the results in Section V.

1. Basic formalism

1.1. Standard elements of the variation procedure

The action functional of the theory, which we consider below, has the standard structure:

$$S = \int d^4x\sqrt{-g}\left\{\frac{R+2\Lambda}{2\kappa} + L_{(\rm total)}\right\} , \tag{1}$$

where g is the determinant of the metric, R is the Ricci scalar, Λ is the cosmological constant and $\kappa = \frac{8\pi G}{c^4}$ is the Einstein constant. The Lagrangian of the physical system as a whole, $L_{(\rm total)}$, can include the metric, pseudoscalar field ϕ and its gradient four-vector $\nabla_k\phi$; it can contain vector field (V^k or U^i) and the covariant derivative ($\nabla_m V^k$ or $\nabla_m U^k$); the Maxwell tensor F_{mn} also can be the constructive element of the Lagrangian; finally, the Ricci and Riemann tensors can appear, when one deals with the non-minimal version of the theory (see, e.g., [44]).

The Einstein field equations appear as the result of variation with respect to metric

$$R_{ik} - \frac{1}{2}g_{ik}R = \Lambda g_{ik} + \kappa T^{(\rm total)}_{ik} , \tag{2}$$

where R_{ik} is the Ricci tensor, and the effective stress-energy tensor $T^{(\rm total)}_{ik}$ has the following *formal* definition

$$T^{(\rm total)}_{ik} \equiv \frac{(-2)}{\sqrt{-g}}\frac{\delta}{\delta g^{ik}}\left[\sqrt{-g}L_{(\rm total)}\right] . \tag{3}$$

This tensor is symmetric by definition and has to be divergence-free due to the Bianchi identities:

$$T^{(\mathrm{total})}_{ik} = T^{(\mathrm{total})}_{ki} \,, \quad \nabla^k T^{(\mathrm{total})}_{ik} = 0 \,. \tag{4}$$

The total Lagrangian of the chiral dynamo-optically active system under consideration can be reconstructed as the sum of four physically distinguished parts

$$L_{(\mathrm{total})} = L_{(\mathrm{em})} + L_{(\mathrm{ps})} + L_{(\mathrm{vect})} + L_{(\mathrm{matter})} \,, \tag{5}$$

associated with the electromagnetic, pseudoscalar, vector fields and matter, respectively. Consider them in more detail.

1.2. Master equations for the electromagnetic field

We assume that the first (electromagnetic) part is quadratic in the Maxwell tensor F_{pq}

$$L_{(\mathrm{em})} = \frac{1}{4} C^{pqmn} F_{pq} F_{mn} \,, \tag{6}$$

and other parts of the Lagrangian do not contain the Maxwell tensor. The Maxwell tensor is the anti-symmetrized derivative of the potential four-vector A_k:

$$F_{mn} \equiv \nabla_m A_n - \nabla_n A_m = \partial_m A_n - \partial_n A_m \,. \tag{7}$$

The definition of the Maxwell tensor provides the first subset of master equations of covariant electrodynamics

$$\nabla_l F_{mn} + \nabla_n F_{lm} + \nabla_m F_{nl} = 0 \,, \tag{8}$$

which can be standardly rewritten in the compact form using the dual tensor F^{*ik}:

$$F^{*ik} \equiv \frac{1}{2} \epsilon^{ikmn} F_{mn} \quad \Rightarrow \nabla_k F^{*ik} = 0 \,. \tag{9}$$

Here $\epsilon^{ikmn} = \frac{E^{ikmn}}{\sqrt{-g}}$ is the Levi-Civita (pseudo) tensor based on the absolutely skew-symmetric symbol E^{ikmn} ($E^{0123} = 1$). The linear response tensor C^{ikmn} possesses the evident symmetry of indices

$$C^{pqmn} = -C^{qpmn} = C^{mnpq} = -C^{pqnm} \,. \tag{10}$$

We assume that the tensor C^{pqmn} can depend, first, on pseudoscalar field ϕ, second, on the vector field V^i or U^i; third, linearly on the gradient four-vector $\nabla_k \phi$, fourth, linearly on the covariant derivative $\nabla_k V^i$ or $\nabla_k U^i$. Such assumptions allow us to describe the interactions between electromagnetic field and pseudoscalar field, on the one hand, and the coupling of the electromagnetic and vector fields. Being the tensor quantity, C^{pqmn} can include the metric, Kronecker deltas, Levi-Civita tensor, as well as, the Riemann, Ricci tensors and Ricci scalar, if one deals with the non-minimal theory.

The second subset of the master equations for the electromagnetic field can be standardly obtained by variation of the action functional with respect to the potential four-(co)vector A_i. This procedure yields

$$\nabla_k \left[C^{ikmn} F_{mn} \right] = -\frac{4\pi}{c} J^i \,, \tag{11}$$

where the four-vector J^i is the electric current defined formally as

$$J^i \equiv \frac{1}{4\pi} \frac{\delta L_{(\mathrm{matter})}}{\delta A_i} \,. \tag{12}$$

It is convenient to use the skew-symmetric induction tensor H^{ik} defined as

$$H^{ik} \equiv C^{ikmn} F_{mn} \,, \tag{13}$$

which is the divergence - free one, when the medium is non-conducting, i.e., $J^i = 0$.

1.3. Master equation for the pseudoscalar field

We assume that the second (pseudoscalar) part of the Lagrangian is quadratic in the gradient four-vector $\nabla_k \phi$, and contain the dimensionless pseudoscalar field ϕ in even combinations

$$L_{(\mathrm{ps})} = \frac{1}{2}\Psi_0^2 \left[-\mathcal{C}^{mn} \nabla_m \phi \nabla_n \phi + \mathcal{V}(\phi^2) \right] . \tag{14}$$

The constitutive tensor $\mathcal{C}^{mn}$ is assumed to depend on the metric, Kronecker deltas, Levi-Civita (pseudo) tensor, and on the velocity and its covariant derivative. $\mathcal{V}(\phi^2)$ is the potential of the pseudoscalar field; the parameter Ψ_0 is reciprocal to the axion-photon coupling constant $\frac{1}{\Psi_0} = g_{A\gamma\gamma}$. Master equations for the pseudoscalar field have the form

$$\nabla_m \left[\mathcal{C}^{mn} \nabla_n \phi \right] + \phi \mathcal{V}'(\phi^2) = \mathcal{J} \,, \tag{15}$$

where the pseudoscalar source is explicitly quadratic in the Maxwell tensor

$$\mathcal{J} = -\frac{1}{4\Psi_0^2} F_{pq} F_{mn} \frac{\partial}{\partial \phi} C^{pqmn} + \frac{1}{4\Psi_0^2} \nabla_j \left[F_{pq} F_{mn} \frac{\partial}{\partial (\nabla_j \phi)} C^{pqmn} \right] , \tag{16}$$

and can depend on the vector field and its covariant derivative, when the linear response tensor C^{pqmn} is correspondingly extended.

1.4. Master equations for the vector field: I. The tetrad paradigm

The tetrad paradigm assumes that there is no additional part in the Lagrangian, i.e., $L_{(\mathrm{vect})} = 0$, and the velocity four-vector V^i is the eigen-vector of the effective stress-energy tensor $T^{(\mathrm{total})}_{ik}$:

$$T^{(\mathrm{total})}_{ik} V^k = W_{(\mathrm{total})} V_i \,. \tag{17}$$

The vector V^i is assumed to be time-like and unit

$$g_{ik} V^i V^k = 1 \,, \tag{18}$$

so that the corresponding eigen-value $W_{(\mathrm{total})}$

$$W_{(\mathrm{total})} = V^i T^{(\mathrm{total})}_{ik} V^k \tag{19}$$

can be indicated as the energy density scalar. With this definition (it is usually indicated as the Landau-Lifshitz definition) the structure of the effective stress-energy tensor is

$$T^{(\mathrm{total})}_{ik} = W_{(\mathrm{total})} V_i V_k + \mathcal{P}^{(\mathrm{total})}_{ik} \,. \tag{20}$$

Here the tensor $\mathcal{P}^{(\mathrm{total})}_{ik}$ is symmetric, orthogonal to the velocity V^i and describes the total pressure tensor of the system. In this approach the velocity four-vector has to satisfy the master equations, which are derived from the conservation law (4). Indeed, the divergence of of the tensor (20) is equal to zero, when

$$V^k \nabla_k [W_{(\mathrm{total})} V_i] + W_{(\mathrm{total})} V_i (\nabla_k V^k) + \nabla^k \mathcal{P}^{(\mathrm{total})}_{ik} = 0 \,. \tag{21}$$

As usual, the projection of (21) on the direction pointed by the velocity V^i gives the equation of the energy density evolution

$$D W_{(\mathrm{total})} + W_{(\mathrm{total})} \Theta = \mathcal{P}^{(\mathrm{total})}_{ik} \nabla^k V^i \,, \tag{22}$$

where $D \equiv V^k \nabla_k$ is the convective derivative, and $\Theta = \nabla_k V^k$ is the extension scalar of the velocity field. The projection of (21) on the hyper-surface orthogonal to the velocity four-vector yields

$$W_{(\mathrm{total})} D V^s + \Delta^{is} \nabla^k \mathcal{P}^{(\mathrm{total})}_{ik} = 0 \,, \tag{23}$$

where $\Delta^{is} \equiv g^{is} - V^i V^s$ is the projector, which is known to possess the following properties:

$$\Delta^{is} = \Delta^{si} \,, \quad \Delta^{is} V_s = 0 \,, \quad \Delta^s_s = 3 \,, \quad \Delta^{is}\Delta_{js} = \Delta^i_j \,. \tag{24}$$

Thus, the unit time-like velocity four-vector V^i in the tetrad paradigm has to satisfy the equations (23).

The velocity four-vector V^i can be included into the set of tetrad vectors $X^i_{(a)}$; the index (a) takes the values $(0),(1),(2),(3)$, and $X^i_{(0)} \equiv V^i$. This quartet of four-vectors satisfies the orthogonality - normalization conditions

$$g_{ik} X^i_{(a)} X^k_{(b)} = \eta_{(a)(b)} \,, \tag{25}$$

$$\eta^{(a)(b)} X^p_{(a)} X^q_{(b)} = g^{pq} \,, \tag{26}$$

where $\eta_{(a)(b)}$ denotes the Minkowski matrix, diagonal $(1,-1,-1,-1)$. Clearly, the tetrad four-vectors are linked by the relation containing the metric, thus, we have to define the working formulas for the variation $\frac{\delta X^j_{(a)}}{\delta g^{ik}}$. This procedure is described in [17], we recall the main details of this procedure. First, the variation of (26) yields

$$\delta g^{pq} = \eta^{(c)(d)} \left[X^q_{(d)} \delta X^p_{(c)} + X^p_{(c)} \delta X^q_{(d)} \right] \,, \tag{27}$$

thus, we obtain the consequence

$$X^{(a)}_p \delta g^{pq} X^{(b)}_q = \left[X^{(a)}_p \delta X^p_{(c)} \eta^{(c)(b)} + \delta X^q_{(d)} X^{(b)}_q \eta^{(a)(d)} \right] \,. \tag{28}$$

Second, the variation $\delta X^i_{(a)}$ can be decomposed as the linear combination of the tetrad four-vectors:

$$\delta X^i_{(a)} = X^i_{(f)} Y_{(a)}^{\ \ (f)} \,. \tag{29}$$

If we put (29) into (28) we obtain

$$Y^{(a)(b)} + Y^{(b)(a)} = \delta g^{pq} X^{(a)}_p X^{(b)}_q \,. \tag{30}$$

Generally, the object $Y^{(a)(b)}$ has the symmetric and antisymmetric parts, $Y^{(a)(b)} = Y^{((a)(b))} + Y^{[(a)(b)]}$, however, only the symmetric part is assumed to be formed by the metric variation; this idea gives immediately that

$$\delta X^i_{(a)} = \frac{1}{4} \delta g^{pq} \left[X_{p(a)} \delta^i_q + X_{q(a)} \delta^i_p \right] \,, \tag{31}$$

and consequently, for the velocity four-vector we have

$$\frac{\delta V^i}{\delta g^{pq}} = \frac{1}{4} \left[V_p \delta^i_q + V_q \delta^i_p \right] \,, \quad \frac{\delta V_i}{\delta g^{pq}} = -\frac{1}{4} \left[V_p g_{iq} + V_q g_{ip} \right] \,. \tag{32}$$

When the linear response tensor C^{ikmn} depends on the covariant derivative of the velocity four-vector, we need to prepare the formula for variation of $\nabla_m V^l$:

$$\delta [\nabla_m V^l] = \nabla_m (\delta V^l) + V^n \delta \Gamma^l_{mn} =$$

$$= \frac{1}{4} \delta g^{pq} \left(\delta^l_p \nabla_m V_q + \delta^l_q \nabla_m V_p \right) + \frac{1}{4} \left(V_p g_{mq} + V_q g_{mp} \right) \nabla^l \delta g^{pq} - \frac{1}{4} \left(\delta^l_p g_{mq} + \delta^l_q g_{mp} \right) V^n \nabla_n \delta g^{pq} \,. \tag{33}$$

Clearly, it contains the terms of the type $\nabla_n \delta g^{pq}$, and thus the variation procedure requires the corresponding integration by part, when we calculate the stress-energy tensor of the electromagnetic field.

1.5. Master equations for the vector field: II. The aether paradigm

The aether paradigm assumes that there exist an additional time-like vector field U^i, and it has to be included into variation procedure as an independent player. To be more precise, the corresponding part of the Lagrangian is non-vanishing

$$L_{(\mathrm{vect})} = \frac{1}{2\kappa}\left[\lambda(g_{pq}U^pU^q - 1) + K^{ab}{}_{mn}\nabla_a U^m \nabla_b U^n\right] , \tag{34}$$

the function λ is the Lagrange multiplier providing the vector field to be normalized by unity; the Jacobson's constitutive tensor $K^{ab}{}_{mn}$ is of the form

$$K^{ab}{}_{mn} = C_1 g^{ab} g_{mn} + C_2 \delta^a_m \delta^b_n + C_3 \delta^a_n \delta^b_m + C_4 U^a U^b g_{mn} , \tag{35}$$

where C_1, C_2, C_3 and C_4 are the phenomenological parameters (see, e.g., [34]). The term (34) is the participant of three variation procedures. First, the variation with respect to the Lagrange multiplier λ yields $g_{pq}U^pU^q = 1$, i.e., the vector field is normalized by unity, and thus it is time-like everywhere; these properties support the idea to consider this vector field as the one of a global velocity. Second, the variation of the total action functional with respect to the vector field U^i gives following equation:

$$\frac{\lambda}{\kappa}U_j - \frac{1}{\kappa}\nabla_a\left[K^{ab}{}_{jn}\nabla_b U^n\right] + \frac{1}{\kappa}C_4 \nabla_j U^n U^b \nabla_b U_n + \frac{1}{4}F_{pq}F_{mn}\frac{\partial C^{pqmn}}{\partial U^j} - \frac{1}{4}\nabla_l\left[F_{pq}F_{mn}\frac{\partial C^{pqmn}}{\partial(\nabla_l U^j)}\right] -$$
$$- \frac{1}{2}\Psi_0^2 \nabla_m\phi \nabla_n\phi\ \frac{\partial C^{mn}}{\partial U^j} + \frac{1}{2}\Psi_0^2\nabla_l\left[\nabla_m\phi\nabla_n\phi\ \frac{\partial C^{mn}}{\partial(\nabla_l U^j)}\right] = 0 . \tag{36}$$

This equation can be rewritten in the well-known form

$$\nabla_a \mathcal{J}^a_j = I^{(\mathrm{U})}_j + \kappa I^{(\mathrm{F})}_j + \kappa I^{(\phi)}_j + \lambda\ U_j , \tag{37}$$

where the following definitions are used:

$$\mathcal{J}^a_j = K^{ab}{}_{jn}(\nabla_b U^n) , \quad I^{(\mathrm{U})}_j = C_4 \nabla_j U^n U^b \nabla_b U_n , \tag{38}$$

$$I^{(\mathrm{F})}_j = \frac{1}{4}F_{pq}F_{mn}\ \frac{\partial C^{pqmn}}{\partial U^j} - \frac{1}{4}\nabla_l\left[F_{pq}F_{mn}\ \frac{\partial C^{pqmn}}{\partial(\nabla_l U^j)}\right] , \tag{39}$$

$$I^{(\phi)}_j = -\frac{1}{2}\Psi_0^2\nabla_m\phi\nabla_n\phi\ \frac{\partial C^{mn}}{\partial U^j} + \frac{1}{2}\Psi_0^2\nabla_l\left[\nabla_m\phi\nabla_n\phi\ \frac{\partial C^{mn}}{\partial(\nabla_l U^j)}\right] . \tag{40}$$

Clearly, using the projection of the equation (37) on the direction U^j and the normalization condition we can obtain the Lagrange multiplier

$$\lambda = U^j\left[\nabla_a\mathcal{J}^a_j - I^{(\mathrm{U})}_j - \kappa I^{(\mathrm{F})}_j - \kappa I^{(\phi)}_j\right] . \tag{41}$$

As well, using the projector $\Delta^{ik} = g^{ik} - U^iU^k$, we can obtain the equation

$$\Delta^{sj}\nabla_a \mathcal{J}^a_j = \Delta^{sj}\left[I^{(\mathrm{U})}_j + \kappa I^{(\mathrm{F})}_j + \kappa I^{(\phi)}_j\right] , \tag{42}$$

which includes the velocity four-vector but does not contain the Lagrange multiplier.

1.6. Standard auxiliary tensor quantities and their interpretation

1.6.1 Decomposition of the covariant derivative of the velocity four-vector

The covariant derivative ∇_k is known to be presented as the decomposition on the longitudinal and transversal components with respect to chosen velocity four-vector; when we deal with the vector field U^i, we have, respectively:

$$\nabla_k = U_i D U_k + \overset{\perp}{\nabla}_k , \quad D \equiv U^m\nabla_m , \quad \overset{\perp}{\nabla}_k \equiv \Delta^m_k \nabla_m , \quad \Delta^m_k \equiv \delta^m_k - U^m U_k , \tag{43}$$

where D is the convective derivative, and Δ^m_k is the projector. In these terms the tensor $\nabla_i U_k$ can be represented as follows:

$$\nabla_i U_k = U_i D U_k + \sigma_{ik} + \omega_{ik} + \frac{1}{3}\Delta_{ik}\Theta \,, \tag{44}$$

where DU^i is the acceleration four-vector, σ_{ik} is the symmetric trace-free shear tensor, ω_{ik} is the skew-symmetric vorticity tensor, and Θ is the expansion scalar. The definitions of these quantities are well-known

$$DU_k \equiv U^m \nabla_m U_k \,, \quad \sigma_{ik} \equiv \frac{1}{2}\left(\overset{\perp}{\nabla}_i U_k + \overset{\perp}{\nabla}_k U_i\right) - \frac{1}{3}\Delta_{ik}\Theta \,,$$

$$\omega_{ik} \equiv \frac{1}{2}\left(\overset{\perp}{\nabla}_i U_k - \overset{\perp}{\nabla}_k U_i\right) \,, \quad \Theta \equiv \nabla_m U^m = \overset{\perp}{\nabla}_m U^m \,. \tag{45}$$

The terms acceleration, shear, vorticity and expansion relate in this case to the aether flow. When we deal with the velocity four-vector V^i, the decomposition is similar.

1.6.2 Decomposition of the Maxwell tensor F_{ik} and of the induction tensor H^{mn}

Electrodynamics of continuous media operates with the quartet of four-vectors D^i, E^i, H^i and B^i. When one deals with the velocity four-vector V^k, these quantities are defined as follows:

$$D^i \equiv H^{ik} V_k \,, \quad H_i \equiv H^*_{ik} V^k \,, \quad E^i \equiv F^{ik} V_k \,, \quad B_i \equiv F^*_{ik} V^k \,. \tag{46}$$

When we work in the aether paradigm, we have to replace V^k with U^k. The four-vectors D^i, E^i, H^i and B^i are orthogonal to the corresponding velocity four-vector. In these terms the tensors F_{ik}, F^*_{ik}, H^{ik} and H^{*ik} can be represented as follows:

$$F_{ik} = E_i V_k - E_k V_i - \epsilon_{ikmn} B^m V^n \,, \quad F^*_{ik} = B_i V_k - B_k V_i + \epsilon_{ikmn} E^m V^n \,, \tag{47}$$

$$H^{ik} = D^i V^k - D^k V^i - \epsilon^{ikmn} H_m V_n \,, \quad H^{*ik} = H^i V^k - H^k V^i + \epsilon^{ikmn} D_m V_n \,. \tag{48}$$

E^i can be interpreted as the four-vector of electric field found in the frame of reference associated with the velocity four-vector V^m. B^i describes the magnetic induction, D^i corresponds to the electric induction, H_i can be indicated as the four-vector of the magnetic field.

1.6.3 Decomposition of the linear response tensor

The tensor C^{ikmn} symmetric with respect to the pair index transposition $C^{mnik} = C^{ikmn}$, also can be decomposed using the appropriate vector field; when we deal with the four-vector V^k the corresponding decomposition is (see, e.g., [1,2] for details):

$$C^{ikmn} = \frac{1}{2}\left[\varepsilon^{im} V^k V^n - \varepsilon^{in} V^k V^m + \varepsilon^{kn} V^i V^m - \varepsilon^{km} V^i V^n\right] -$$

$$- \frac{1}{2}\eta^{ikl}(\mu^{-1})_{ls}\eta^{mns} - \frac{1}{2}\left[\eta^{ikl}(V^m {\nu_l}^n - V^n {\nu_l}^m) + \eta^{lmn}(V^i {\nu_l}^k - V^k {\nu_l}^i)\right] \,. \tag{49}$$

The new two-indices tensors are defined as follows:

$$\varepsilon^{im} = 2C^{ikmn} V_k V_n \,, \quad (\mu^{-1})_{pq} = -\frac{1}{2}\eta_{pik} C^{ikmn} \eta_{mnq} \,, \quad {\nu_p}^m = \eta_{pik} C^{ikmn} V_n \,, \tag{50}$$

where $\eta_{pik} \equiv \epsilon_{pikq} V^q$. The tensors ε_{ik} and $(\mu^{-1})_{ik}$ are symmetric, ${\nu_l}^k$ is, in general, non-symmetric; they are orthogonal to V^i, i.e.

$$\varepsilon_{ik} V^k = 0 \,, \quad (\mu^{-1})_{ik} V^k = 0 \,, \quad {\nu_l}^k V^l = 0 = {\nu_l}^k V_k \,. \tag{51}$$

The tensor ε_{ik} is interpreted as the dielectric permeability tensor found in the frame of reference associated with the velocity four-vector V^i; the tensor $(\mu^{-1})_{ik}$ describes the magnetic impermeability of the medium; the tensor ${\nu_i}^k$ contains the so-called magneto-electric coefficients of the medium. This interpretation is based on the formula

$$D^i = \epsilon^{ik} E_k - {\nu_k}^i B^k \,, \quad H_i = {\nu_i}^k E_k + (\mu^{-1})_{ik} B^k \,, \tag{52}$$

which can be directly obtained using the definitions presented above.

2. Effective stress-energy tensor of the electromagnetic field in a dynamo-optically active medium

A number of details of the variation formalism based on the tetrad and aether paradigms coincide. For instance, the following auxiliary variational identities are of common use:

$$\frac{\delta}{\delta g^{ik}}\phi = 0\,, \quad \frac{\delta}{\delta g^{ik}}\nabla_m\phi = 0\,, \quad \frac{\delta}{\delta g^{ik}}F_{mn} = 0\,, \quad \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{ik}} = -\frac{1}{2}g_{ik}\,, \tag{53}$$

$$\frac{\delta g_{ls}}{\delta g^{ik}} = -\frac{1}{2}\left[g_{li}g_{ks} + g_{lk}g_{is}\right]\,, \quad \frac{\delta}{\delta g^{ik}}\delta^q_p = 0\,, \quad \frac{\delta \epsilon^{lsrt}}{\delta g^{ik}} = \frac{1}{2}\epsilon^{lsrt}g_{ik}\,. \tag{54}$$

However, all the details of procedures, which relate to variation with respect to velocity four-vector and its covariant derivative, have to be considered individually, if we follow tetrad or aether paradigms.

2.1. Calculations in the framework of the tetrad paradigm

In the framework of the tetrad formalism we use the following definition of the electromagnetic stress-energy tensor:

$$T^{(\rm em)}_{ik} = -\frac{1}{2\sqrt{-g}}\frac{\delta}{\delta g^{ik}}\left[\sqrt{-g}F_{pq}F_{mn}C^{pqmn}\right]\,. \tag{55}$$

Taking into account (53) we obtain immediately

$$T^{(\rm em)}_{ik} = \frac{1}{4}F_{pq}F_{mn}C^{pqmn} - \frac{1}{2}F_{pq}F_{mn}\frac{\delta}{\delta g^{ik}}C^{pqmn}\,. \tag{56}$$

The first term is the scalar $\frac{1}{4}H^{mn}F_{mn}$, which is the part of all known stress-energy tensors of the electromagnetic field in media; the difference between them appears due to the second term. Keeping in mind (54) we can rewrite (56) as follows:

$$T^{(\rm em)}_{ik} = \frac{1}{4}F_{pq}F_{mn}\left\{g_{ik}\left[C^{pqmn} - \epsilon^{lsrt}\,\frac{\partial C^{pqmn}}{\partial \epsilon^{lsrt}}\right] - \left(\delta^l_i\delta^s_k + \delta^l_k\delta^s_i\right)\,\frac{\partial C^{pqmn}}{\partial g^{ls}} - \right.$$
$$\left. -\frac{1}{2}\left(V_i\delta^j_k + V_k\delta^j_i\right)\,\frac{\partial C^{pqmn}}{\partial V^j} - \frac{1}{2}\left(\delta^j_i\nabla_l V_k + \delta^j_k\nabla_l V_i\right)\,\frac{\partial C^{pqmn}}{\partial(\nabla_l V^j)}\right\} +$$
$$+\frac{1}{8}\nabla^j\left[\left(V_i g_{lk} + V_k g_{li}\right)F_{pq}F_{mn}\frac{\partial C^{pqmn}}{\partial(\nabla_l V^j)}\right] - \frac{1}{8}\left(\delta^j_i g_{lk} + \delta^j_k g_{li}\right)\nabla_s\left[F_{pq}F_{mn}V^s\frac{\partial C^{pqmn}}{\partial(\nabla_l V^j)}\right]\,. \tag{57}$$

One has to stress that we deal with the example of the theory, in which the stress-energy tensor contains not only the Maxwell tensor, but its covariant derivative $\nabla_s F_{mn}$ also, since the linear response tensor C^{pqmn} is assumed to contain the dynamo-optical terms, i.e., since $\frac{\partial C^{pqmn}}{\partial(\nabla_l V^j)} \neq 0$.

2.2. Calculations in the framework of the aether paradigm

Now we consider the vector field U^j to be independent on the variation of the metric, i.e., in contrast to (32), we have $\frac{\delta U^j}{\delta g^{ik}} = 0$. Also, we keep in mind, that the variation of the term $\frac{1}{2}\lambda(g_{mn}U^mU^n - 1)$ with respect to metric g^{ik} gives the contribution $\lambda U_i U_k$ into the total stress-energy tensor. The quantity λ given by (41) contains the part $I^{(\rm F)}_j$, which according to (39) is quadratic in the Maxwell tensor; we add this term to the stress-energy tensor of the electromagnetic field. The variation of the covariant derivative also differ from (33), being of the following form:

$$\delta(\nabla_l U^j) = -\frac{1}{2}\left[\delta^j_{(i}g_{k)l}U^n\nabla_n + \delta^j_{(i}U_{k)}\nabla_l - U_{(i}g_{k)l}\nabla^j\right]\delta g^{ik}\,. \tag{58}$$

We use here and below the standard definition of the symmetrization: $A_{(i}B_{k)} \equiv \frac{1}{2}(A_iB_k{+}A_kB_i)$. Now the stress-energy tensor of the electromagnetic field can be written in the following form

$$\mathcal{T}^{(\rm em)}_{ik} = \frac{1}{4}F_{pq}F_{mn}\left\{g_{ik}\left[C^{pqmn} - \epsilon^{lsrt}\ \frac{\partial C^{pqmn}}{\partial \epsilon^{lsrt}}\right] - (\delta^l_i\delta^s_k + \delta^l_k\delta^s_i)\ \frac{\partial C^{pqmn}}{\partial g^{ls}}\right\} +$$

$$-\frac{1}{4}U_iU_kU^j\left\{F_{pq}F_{mn}\ \frac{\partial C^{pqmn}}{\partial U^j} - \nabla_l\left[F_{pq}F_{mn}\ \frac{\partial C^{pqmn}}{\partial(\nabla_l U^j)}\right]\right\} -$$

$$-\frac{1}{8}\nabla_l\left[\left(\delta^j_iU_k + \delta^j_kU_i\right)\ F_{pq}F_{mn}\frac{\partial C^{pqmn}}{\partial(\nabla_l U^j)}\right] +$$

$$+\frac{1}{8}\nabla^j\left[(U_ig_{lk} + U_kg_{li})\,F_{pq}F_{mn}\frac{\partial C^{pqmn}}{\partial(\nabla_l U^j)}\right] - \frac{1}{8}\left(\delta^j_i g_{lk} + \delta^j_k g_{li}\right)\nabla_n\left[F_{pq}F_{mn}U^n\frac{\partial C^{pqmn}}{\partial(\nabla_l U^j)}\right]. \tag{59}$$

Let us recall how to reconstruct the basic (irreducible) elements of the stress-energy tensor of the electromagnetic field.

2.3. Energy density, energy flux four-vector and the pressure tensor of the electromagnetic field: Do they differ in the tetrad and aether paradigms?

The standard decomposition of the symmetric effective stress-energy tensor contains three basic elements: the energy density scalar W, the flux four-vector $\mathcal{Q}^k$ and the pressure tensor $\mathcal{P}^{ik}$. In the framework of the tetrad paradigm they are defined, respectively, as

$$W \equiv V^m T^{(\rm em)}_{mn} V^n\,, \tag{60}$$

$$\mathcal{Q}^k \equiv V^m T^{(\rm em)}_{mn}\Delta^{kn} = \Delta^{km}T^{(\rm em)}_{mn}V^n\,, \tag{61}$$

$$\mathcal{P}^{ik} \equiv \Delta^{im}T^{(\rm em)}_{mn}\Delta^{kn}\,. \tag{62}$$

In order to obtain the corresponding quantities in the framework of the aether paradigm we have to replace U^j with V^j and $T^{(\rm em)}_{mn}$ with $\mathcal{T}^{(\rm em)}_{mn}$. We are interested to calculate the difference

$$\tau_{ik} \equiv \mathcal{T}^{(\rm em)}_{ik} - T^{(\rm em)}_{ik}\,. \tag{63}$$

When $U^j = V^j$, we obtain immediately that τ_{ik} is of the form:

$$\tau_{ik} = \frac{1}{4}U_{(i}\Delta^j_{k)}\left\{F_{pq}F_{mn}\frac{\partial C^{pqmn}}{\partial U^j} - \nabla_l\left[F_{pq}F_{mn}\frac{\partial C^{pqmn}}{\partial(\nabla_l U^j)}\right]\right\}. \tag{64}$$

Clearly, the energy density scalars and pressure tensors, calculated using the tetrad and aether paradigms, coincide:

$$U^i\tau_{ik}U^k = 0\ \Rightarrow W^{(\rm aether)} = W^{(\rm tetrad)}\,, \tag{65}$$

$$\Delta^i_m\tau_{ik}\Delta^k_n = 0\ \Rightarrow \mathcal{P}^{(\rm aether)}_{mn} = \mathcal{P}^{(\rm tetrad)}_{mn}\,. \tag{66}$$

Only the flux four-vectors differ:

$$\mathcal{Q}^h_{(\rm aether)} - \mathcal{Q}^h_{(\rm tetrad)} = \Delta^{ih}\tau_{ik}U^k = \frac{1}{8}\Delta^{jh}\left\{F_{pq}F_{mn}\frac{\partial C^{pqmn}}{\partial U^j} - \nabla_l\left[F_{pq}F_{mn}\frac{\partial C^{pqmn}}{\partial(\nabla_l U^j}\right]\right\}. \tag{67}$$

Thus, we have to return to the Minkowski-Abraham controversy and to discuss this difference. Let us consider three model before starting to analyze the problem.

3. Three examples of the linear response tensor

3.1. Axionic vacuum

In the first example the linear response tensor is assumed to contain neither velocity four-vector, nor its covariant derivative:

$$C^{pqmn}_{(\rm vacuum)} = \frac{1}{2}\left(g^{pqmn} + \phi\ \epsilon^{pqmn}\right) . \tag{68}$$

Here and below we use the auxiliary tensor

$$g^{pqmn} \equiv g^{pm}g^{qn} - g^{pn}g^{qm} . \tag{69}$$

Using the definitions (50) we obtain

$$\varepsilon^{im} = \Delta^{im} , \quad (\mu^{-1})_{pq} = \Delta_{pq} , \quad {\nu_p}^m = -\phi\Delta^m_p . \tag{70}$$

Since ${\nu_p}^m \neq 0$, this medium possesses magnetoelectric properties, which are provided by the presence of the pseudoscalar field ϕ. Calculations in both: tetrad and aether paradigms (see (57) and (59), respectively), give the same traceless tensor

$$T^{(\rm vacuum)}_{ik} = \mathcal{T}^{(\rm vacuum)}_{ik} = \frac{1}{4}g_{ik}F_{mn}F^{mn} - F_{im}{F_k}^m . \tag{71}$$

In other words, the stress-energy tensors do not differ one from another, and they do not contain axionic field. Respectively, the energy density scalars, energy flux four-vectors and pressure tensors

$$W = -\frac{1}{2}\left(E^m E_m + B^m B_m\right) , \quad \mathcal{Q}^j = -\eta^{jmn}E_m B_n ,$$

$$\mathcal{P}^{pq} = \frac{1}{2}\Delta^{pq}\left(E^m E_m + B^m B_m\right) - \left(E^p E^q + B^p B^q\right) . \tag{72}$$

formally coincide for both definitions of the velocity four vector, V^i and U^i.

3.2. Spatially isotropic homogeneous moving dielectric medium

3.2.1 Calculations in the context of the tetrad paradigm

The linear response tensor contains now terms quadratic in the velocity four-vector:

$$C^{pqmn} = C^{pqmn}_{(0)} + C^{pqmn}_{(\phi)} , \tag{73}$$

$$C^{pqmn}_{(0)} = \frac{1}{2\mu}\left[(g^{pm}g^{qn} - g^{pn}g^{qm}) + (\varepsilon\mu - 1)\left(g^{pm}V^qV^n - g^{pn}V^qV^m + g^{qn}V^pV^m - g^{qm}V^pV^n\right)\right] , \tag{74}$$

$$C^{pqmn}_{(\phi)} \equiv \frac{1}{2}\phi\left[\epsilon^{pqmn} + \nu g_{rh}V^h\left(V^p\epsilon^{rqmn} - V^q\epsilon^{rpmn} + V^m\epsilon^{rnpq} - V^n\epsilon^{rmpq}\right)\right] . \tag{75}$$

Using the definitions (50) we again calculate the permittivity tensors and the tensor of magneto-electric coefficients:

$$\varepsilon^{im} = \varepsilon\Delta^{im} , \quad (\mu^{-1})_{pq} = \frac{1}{\mu}\Delta_{pq} , \quad {\nu_p}^m = -\phi\Delta^m_p(1+\nu) . \tag{76}$$

Thus, ε characterizes the dielectric permittivity; μ is the constant of magnetic permeability; $n = \sqrt{\varepsilon\mu}$ is the refraction index; ν is the magnetoelectric constant. When $\varepsilon = 1$, $\mu = 1$, $\nu = 0$, the tensor C^{pqmn} converts into $C^{pqmn}_{(\rm vacuum)}$ (68). The stress-energy tensor calculated using (57) can be presented in two forms. The first representation contains the Maxwell tensor:

$$T_{ik}^{(\text{isotropic})} = \frac{1}{4} g_{ik} F_{pq} F_{mn} C_{(0)}^{pqmn} - \frac{1}{2} \left[g_{pi} F_{kq} + g_{pk} F_{iq} \right] C_{(0)}^{pqmn} F_{mn} \,. \tag{77}$$

The term $C_{(\phi)}^{pqmn}$ disappears from the stress-energy tensor of the electromagnetic field due to the relations (32), and due to the identity

$$F^{im} F^{*}_{km} = \frac{1}{4} \delta^i_k F^{mn} F^{*}_{mn} \,. \tag{78}$$

The second form of the stress-energy tensor contains the four-vectors E^i and B^k:

$$T_{ik}^{(\text{isotropic})} = \left(\frac{1}{2} g_{ik} - V_i V_k \right) \left(\varepsilon E^m E_m + \frac{1}{\mu} B^m B_m \right) - \left(\varepsilon E_i E_k + \frac{1}{\mu} B_i B_k \right) -$$

$$- \frac{1}{2} \left(\varepsilon + \frac{1}{\mu} \right) \left(V_i \eta_{kmn} + V_k \eta_{imn} \right) E^m B^n \,. \tag{79}$$

Clearly, the tensor (79) is traceless, and it contains neither the parameter ν, nor the pseudoscalar (axion) field. The formulas

$$W = -\frac{1}{2} \left(\varepsilon E^m E_m + \frac{1}{\mu} B^m B_m \right) \,, \quad \mathcal{Q}^j = -\frac{1}{2} \left(\varepsilon + \frac{1}{\mu} \right) \eta^{jmn} E_m B_n \,,$$

$$\mathcal{P}^{pq} = \frac{1}{2} \Delta^{pq} \left(\varepsilon E^m E_m + \frac{1}{\mu} B^m B_m \right) - \left(\varepsilon E^p E^q + \frac{1}{\mu} B^p B^q \right) \tag{80}$$

describe the energy density of the electromagnetic field, energy flux four-vector and pressure tensor, respectively, when $\epsilon \neq 1$, $\mu \neq 1$, $\nu \neq 0$.

3.2.2 Calculations in the context of the aether paradigm

Calculations based on the formula (59) yields the following stress-energy tensor:

$$\mathcal{T}_{ik}^{(\text{isotropic})} = \left(\frac{1}{2} g_{ik} - U_i U_k \right) \left(\varepsilon E^m E_m + \frac{1}{\mu} B^m B_m \right) - \left(\varepsilon E_i E_k + \frac{1}{\mu} B_i B_k \right) -$$

$$- \frac{1}{\mu} \left(U_i \eta_{kmn} + U_k \eta_{imn} \right) E^m B^n \,. \tag{81}$$

Clearly, the corresponding energy-density scalar and the pressure tensor

$$\mathcal{W} = -\frac{1}{2} \left(\varepsilon E^m E_m + \frac{1}{\mu} B^m B_m \right) \,, \tag{82}$$

$$\mathcal{P}^{pq} = \frac{1}{2} \Delta^{pq} \left(\varepsilon E^m E_m + \frac{1}{\mu} B^m B_m \right) - \left(\varepsilon E^p E^q + \frac{1}{\mu} B^p B^q \right) \tag{83}$$

coincide with the ones obtained in the framework of the tetrad paradigm. However, the energy flux four-vector

$$\mathcal{Q}^j = -\frac{1}{\mu} \eta^{jmn} E_m B_n \tag{84}$$

differs from the one given by (80) by the constant multiplier $\frac{1}{2}(n^2+1)$, which is in evident concordance with (67).

3.3. Dynamo-optically active medium

We work in the linear electrodynamics of the chiral (quasi)medium, i.e., adding a new sophisticated element into the linear response tensor C^{pqmn} we obtain a new additional term in the corresponding stress-energy tensor. That is why, as the third example, we consider the model with the linear response tensor, which is simplified to have the following form in the framework of the tetrad paradigm:

$$C^{pqmn} = \frac{1}{2}g^{pqmn} + X^{lspqmn} g_{js} \nabla_l V^j \,. \tag{85}$$

When we deal with the aether paradigm, we have to replace V^i with U^i. In other words, we consider the dynamo-optically active vacuum with ε=1, μ=1, ν=0. The new constitutive tensor

$$X^{lspqmn} = \frac{1}{4} g_{ar} V^r g_{bt} V^t \left[\alpha\left(g^{pqla}g^{mnsb} + g^{mnla}g^{pqsb}\right) - \gamma\left(\epsilon^{pqla}\epsilon^{smnb} + \epsilon^{pqsa}\epsilon^{lmnb}\right)\right] \tag{86}$$

is assumed to contain two new coupling constants α and γ (see [30] for the complete representation of this constitutive tensor). In order to interpret these coupling constants, we calculate the tensors ε^{ik}, $\left(\mu^{-1}\right)^{ik}$ and ν^{ik}, and obtain that

$$\varepsilon^{ik} = \Delta^{ik} + \alpha \overset{\perp}{\nabla}{}^{(i}V^{k)} \,, \quad \left(\mu^{-1}\right)^{ik} = \Delta^{ik} + \gamma \overset{\perp}{\nabla}{}^{(i}V^{k)} \,, \quad \nu^{ik} = 0 \,. \tag{87}$$

Thus, the parameter α is associated with the dynamo-optically induced dielectric susceptibility, while γ relates to the dynamo-optically induced magnetic susceptibility. Now we are ready for calculations of the stress-energy tensor components.

3.3.1 Analysis based on the tetrad paradigm

We use the already obtained tensor (71) and present the whole stress-energy tensor in the following tentative form:

$$\begin{gathered}
T^{(\text{dynamo})}_{ik} - T^{(\text{vacuum})}_{ik} = \\
= \frac{1}{4}F_{pq}F_{mn}\left\{ g_{ik}(\nabla_l V_s)\left[X^{lspqmn} - \epsilon^{fhrt}\frac{\partial X^{lspqmn}}{\partial \epsilon^{fhrt}}\right] - \left(\delta^l_i\delta^s_k + \delta^l_k\delta^s_i\right)\left(\nabla_h V^j\right)\frac{\partial\left(X^{hfpqmn}g_{jf}\right)}{\partial g^{ls}} - \right. \\
\left. - \frac{1}{2}\left(V_i\delta^j_k + V_k\delta^j_i\right)(\nabla_l V_s)\frac{\partial X^{lspqmn}}{\partial V^j} - \frac{1}{2}\left(g_{is}\nabla_l V_k + g_{ks}\nabla_l V_i\right)\ X^{lspqmn}\right\} + \\
+ \frac{1}{8}\nabla_h\left\{F_{pq}F_{mn}\left[\left(V_i g_{lk} + V_k g_{li}\right)X^{lhpqmn} - \left(g_{is}g_{lk} + g_{ks}g_{li}\right)V^h X^{lspqmn}\right]\right\} \,,
\end{gathered} \tag{88}$$

where the tensor X^{lspqmn} is given by (86). Further routine but cumbersome calculations give the following result:

$$\begin{gathered}
T^{(\text{dynamo})}_{ik} = T^{(\text{vacuum})}_{ik} + \\
+ \left(\frac{1}{2}g_{ik} - V_iV_k\right)\left(\alpha E^lE^s + \gamma B^lB^s\right)\overset{\perp}{\nabla}_{(l}V_{s)} - \frac{1}{2}\left[\alpha E^lE^s + \gamma B^lB^s\right]\overset{\perp}{\nabla}_l V_{(k}g_{i)s} - \\
- \frac{1}{2}\left[\alpha E^s V_{(i}E_{k)} - \gamma B^s V_{(i}B_{k)}\right] DV_s - \alpha E^j E_{(i}\overset{\perp}{\nabla}_{k)}V_j - \\
- \left[\alpha B^m E^{(s}\eta^{l)}{}_{m(k}V_{i)} - \gamma E^m B^{(s}\eta^{l)}{}_{m(k}V_{i)}\right]\overset{\perp}{\nabla}_l V_s + \\
+ \frac{1}{2}\nabla_h\left\{\alpha E^h V_{(i}E_{k)} - \gamma B^h V_{(i}B_{k)} - V^h\left[\alpha E_iE_k - \gamma B_iB_k\right]\right\} \,.
\end{gathered} \tag{89}$$

We are interested to find the energy flux four-vector associated with this tensor; it is now of the following form:

$$\mathcal{Q}^h_{(\mathrm{tetrad})} \equiv \Delta^{hi} T^{(\mathrm{dynamo})}_{ik} V^k =$$

$$= \eta^{hmn} B_m E_n + \frac{1}{4} \Delta^h_s \overset{\perp}{\nabla}_l \left(\alpha E^l E^s - \gamma B^l B^s \right) + \frac{1}{2} \left[\alpha B^m \eta^h{}_{m(l} E_{s)} - \gamma E^m \eta^h{}_{m(l} B_{s)} \right] \overset{\perp}{\nabla}{}^{(l} V^{s)} . \tag{90}$$

Keeping in mind that according to (45) $\overset{\perp}{\nabla}{}^{(l} V^{s)} = \sigma^{ls} + \frac{1}{3} \Theta \Delta^{ls}$, we can say that the energy flux depends on the shear tensor σ^{ls} and on the expansion scalar Θ of the velocity flow, but it ignores the acceleration and rotation of the dynamo-optically active medium described by the presented model.

3.3.2 Analysis based on the aether paradigm

In order to describe the stress-energy tensor in the framework of the aether paradigm, we use the consequence of the formulas (63) and (64), which now can be written as follows:

$$\mathcal{T}^{(\mathrm{dynamo})}_{ik} - T^{(\mathrm{dynamo})}_{ik} =$$

$$= -\frac{1}{2} U_{(i} \Delta_{k)s} \overset{\perp}{\nabla}_l \left[\alpha E^l E^s - \gamma B^l B^s \right] + \overset{\perp}{\nabla}_l U_s \left[\alpha B^m E^{(s} \eta^{l)}{}_{m(k} U_{i)} + \gamma E^m B^{(s} \eta^{l)}{}_{m(k} U_{i)} \right] . \tag{91}$$

As it was mentioned above, only the flux four-vectors do not coincide for these two approaches, giving the following difference:

$$\mathcal{Q}^h_{(\mathrm{aether})} - \mathcal{Q}^h_{(\mathrm{tetrad})} =$$

$$= -\frac{1}{4} \Delta^h_s \overset{\perp}{\nabla}_l \left(\alpha E^l E^s - \gamma B^l B^s \right) - \frac{1}{2} \eta^h{}_{ml} \left[\alpha E_s B^m + \gamma B_s E^m \right] \overset{\perp}{\nabla}{}^{(l} U^{s)} . \tag{92}$$

This final result is

$$\mathcal{Q}^h_{(\mathrm{aether})} = \eta^{hmn} B_m E_n - \gamma E^m \eta^h{}_{m(l} B_{s)} \overset{\perp}{\nabla}{}^{(l} V^{s)} , \tag{93}$$

i.e., the energy flux four-vector in the dynamo-optically active medium, calculated in the approach, which we indicated as aether paradigm, does not contain the susceptibility parameter α, but includes the parameter γ.

Conclusion

Readers could ask the authors, what is an expediency to follow sophisticated calculations presented above? Are there some applications of the developed formalism? Answering the last question we would like to recall only one fact. The interpretation of the outstanding astronomical event GW170817 / GRB 170817A (see [47]), which is connected with the discovery of gravitational waves and gamma-rays from a binary neutron star merger, is based on the standard model of the electromagnetic wave propagation and the energy transfer. In other words, for the interpretation of this event the standard formula for the electromagnetic energy flux in vacuum was used. Let us imagine now, that the dynamic aether really exists, that this aether is dynamo-optically active, and that the electromagnetic radiation from the binary system propagates indeed inside the dynamic aether. Then we have to use the formula (93) for estimations . Since we keep in mind the cosmological context, we consider the aether flow to possess only the expansion, so that the covariant derivative of the velocity four-vector of the aether has the form $\nabla_i U_k = H(t) \Delta_{ik}$, where $H(t) = \frac{1}{3}\Theta$ is the Hubble function. Then the formula (93) reduces to $\mathcal{Q}^h_{(\mathrm{aether})} = \eta^{hmn} B_m E_n [1 + \gamma H(t)]$, and the energy flux four-vector differs from the Poynting vector by the multiplier $[1 + \gamma H(t)]$. Is it possible to find this multiplier from observations? It is a not easy question, but certainly it is very interesting one, and we hope to return to this problem in a special work.

Acknowledgements

The work was supported by the Program of Competitive Growth of Kazan Federal University.

References

1. Minkowski H. *Nachr. Ges. Wiss. Göttingen*, 1908, 53.
2. Einstein A., Laub J. *Ann. d. Phys.*, 1908, 26, 541.
3. Abraham M. *Rend. Circ. Mat. Palermo*, 1909, 28, 1;
Abraham M. *Rend. Circ. Mat. Palermo*, 1910, 30, 33.
4. Moller C. The Theory of Relativity, Clarendon, Oxford, 1952.
5. de Groot S.R., Mazur P. Non-Equilibrium Thermodynamics, North-Holland, Amsterdam, 1962.
6. Grot R.A. *J. Math. Phys.*, 1970, 11, 109.
7. de Groot S.R., Suttorp L.G. Foundation of Electrodynamics, North-Holland, Amsterdam, 1972.
8. Israel W. *Gen. Relat. Grav.*, 1978, 9, 451.
9. Maugin G.A. *J. Math. Phys.*, 1978, 19, 1206.
10. Skobeltsyn D.V. *Sov. Phys. Uspekhi.* , 1973, 16, 381.
11. Ginzburg V.L. *Sov. Phys. Uspekhi.*, 1973, 16, 434.
12. Robinson F.N.H. *Phys. Rept.*, 1975, 16, 313.
13. Bressan A. Relativistic Theories of Materials, Springer-Verlag, Berlin, 1978.
14. Brevik I. *Phys. Rept.*, 1979, 52, 133.
15. Antoci S., Minich L. *Nuovo Cim.*, 1997, B 112, 991.
16. Obukhov Yu.N., Hehl F.W. *Phys. Lett.*, 2003, A 311, 277.
17. Balakin A.B. *Gravit. Cosmol.*, 2007, 13, 163.
18. Brevik I., Ellingsen S.A. *Ann. Phys.*, 2011, 326, 754.
19. Zhang W.-Z., Zhang P., Wang R.-Q., Liu W.-M., *Phys. Rev. A*, 2012, 85, 053604.
20. Crenshaw M.E. *Ann. Phys.*, 2013, 338, 97-106.
21. Ramos T., Rubilar G.F., Obukhov Yu.N. *J. Opt.*, 2015, 17, 025611.
22. Bliokh K.Y., Bekshaev A.Y., Nori F. *New J. Phys.*, 2017, 19, 123014.
23. Saldanha P.L., Oliveira Filho J.S. *Phys. Rev. A*, 2017, 95, 043804.
24. Silveirinha M.G. *Phys. Rev. A*, 2017, 96, 033831.
25. Brevik I. *Ann. Phys.*, 2017, 377, 10-21.
26. Brevik I. *Phys. Rev. A*, 2018, 98, 043847.
27. Brevik I. *Mod. Phys. Lett. A*, 2018, 33, 1830006.
28. Balakin A.B. *Symmetry*, 2016, 8, 56.
29. Balakin A.B., Dolbilova N.N. *Phys. Rev. D*, 2014, 89, 104012.
30. Balakin A.B., Lemos J.P.S. *Ann. Phys.*, 2014, 350, 454-484.
31. Alpin T.Yu., Balakin A.B. *Int. J. Mod. Phys. D*, 2016, 25, 1650048.
32. Alpin T.Yu., Balakin A.B. *Eur. Phys. J. C*, 2017, 77, 699.
33. Balakin A.B. *Mod. Phys. Lett. A*, 2018, 33, 1850050.
34. Eling C., Jacobson T. *Phys. Rev. D*, 2004, 69, 064005.
35. Jacobson T., Mattingly D. *Phys. Rev. D*, 2004, 70, 024003.
36. Jacobson T. *Proc. of Scince PoSQG-Ph*, 2007, 020.
37. Will C.M., Nordtvedt K. *Astrophys. J.*, 1972, 177, 757.
38. Nordtvedt K., Will C.M. *Astrophys. J.*, 1972, 177, 775.
39. Hellings R.W., Nordtvedt K. *Phys. Rev. D*, 1973, 7, 3593.
40. Liberati S., Maccione L. *Ann. Rev. Nucl. Part. Sci.*, 2009, 59, 245.
41. Kostelecky A., Mewes M. *Phys. Rev. D*, 2009, 80, 015020.
42. Landau L.D., Lifshitz E.M., Pitaevskii L.P. Electrodynamics of Continuous Media, Butterworth Heinemann, Oxford, 1966.
43. Balakin A.B., Shakirzyanov A.F. arXiv:1804.10054.
44. Balakin A.B., Lemos J.P.S. *Class. Quantum Grav.*, 2005, 22, 1867-1880.

45. Eringen A.C., Maugin G.A. Electrodynamics of Continua, Springer-Verlag, New York, 1989.

46. Hehl F.W., Obukhov Yu.N. Foundations of Classical Electrodynamics: Charge, Flux, and Metric, Birkhäuser, Boston, 2003.

47. LIGO Scientific Collaboration, Virgo Collaboration, Fermi Gamma-Ray Burst Monitor, INTEGRAL, *Astrophys. J. Lett.*, 2017, 848 L13.

Авторы

Альпин Тимур Юрьевич, ассистент, кафедра теории относительности и гравитации, институт физики, ФГАОУ ВО Казанский (Приволжский) федеральный университет, ул. Кремлевская, 16а, Казань, 420008, Россия.
E-mail: Timur.Alpin@kpfu.ru

Балакин Александр Борисович, д. ф.-м. н., профессор, кафедра теории относительности и гравитации, институт физики, ФГАОУ ВО Казанский (Приволжский) федеральный университет, ул. Кремлевская, 16а, Казань, 420008, Россия.
E-mail: Alexander.Balakin@kpfu.ru

Просьба ссылаться на эту статью следующим образом:
Альпин Т. Ю., Балакин А. Б. Динамо-оптически активные среды: новые аспекты полемики Минковского-Абрагама // Пространство, время и фундаментальные взаимодействия. 2018. № 4. С. 32—47.

Authors

Alpin Timur Yurievich, Department of General Relativity and Gravitation, Institute of Physics, Kazan (Volga region) Federal University, Kremlevskaya str., 16a, Kazan, Republic of Tatarstan, 420008, Russia.
E-mail: Timur.Alpin@kpfu.ru

Balakin Alexander Borisovich, Doctor of Physical and Mathematical Sciences, Professor, Department of General Relativity and Gravitation, Institute of Physics, Kazan (Volga region) Federal University, Kremlevskaya str., 16a, Kazan, Republic of Tatarstan, 420008, Russia.
E-mail: Alexander.Balakin@kpfu.ru

Please cite this article in English as:
Alpin T. Y., Balakin A. B. Dynamo-optically active media: new aspects of the Minkowski-Abraham controversy. *Space, Time and Fundamental Interactions*, 2018, no. 4, pp. 32–47.