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FRIEDMANN COSMOLOGICAL MODEL WITH NONLINEAR SCALAR FIELD

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***Abstract*:** This paper is a translation of work Ivanov G.G "Friedmann cosmological model with nonlinear scalar field" of 1981 [1].

***Keywords*:** Ivanov G.G., Friedmann cosmological model, nonlinear scalar field.

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Introduction

In this paper it was considered the influence of a real nonlinear cosmological scalar field on the expansion of the Universe in the framework of flat Friedmann cosmological model as an example.

The Lagrangian of nonlinear scalar field has the form:

$$L = \frac{1}{2}\varphi_{,k}\varphi_{,}^{k} - U(\varphi). \tag{1}$$

The corresponding field equation is obtained by varying the action with respect to ϕ

$$\Box\varphi + \frac{dU}{d\varphi} = 0, \tag{2}$$

where

$$\Box \equiv \frac{1}{\sqrt{-\mathrm{g}}}\frac{\partial}{\partial x^k}(\sqrt{-\mathrm{g}}\mathrm{g}^{ki}\frac{\partial}{\partial x^i}).$$

Variation of the action with respect to g_{ik} leads to the canonical energy-momentum tensor

$$T_{ik} = \varphi_{,i}\varphi_{,k} - \left(\frac{1}{2}\varphi_{,p}\varphi^{,p} - U(\varphi)\right)\mathrm{g}_{ik}. \tag{3}$$

Einstein equation with the source (3) can be written as

$$R_{ik} = \varkappa\left[\varphi_{,i}\varphi_{,k} - U(\varphi)\mathrm{g}_{ik}\right]. \tag{4}$$

1. The Friedmann metric and macroscopic characteristics of a scalar field.

Let us take the Friedmann metric in the synchronous reference system where the linear element has the form

$$ds^2 = -e^{2\sigma(T)}(dx^2 + dy^2 + dz^2) + dT^2. \tag{5}$$

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We assume that the cosmological scalar field $\varphi = \varphi(T)$. Choosing the vector field $V^i = \delta_4^i$ as the vector field of observer in the synchronous reference system, we can determine the internal density of the scalar field ρ and the effective pressure p, since in this case the energy-momentum tensor structure coincides with the energy-momentum tensor structure of a perfect fluid.

Before we start the calculation of ρ and p let us write the field equations

$$-\sigma'' = \frac{\varkappa}{2}\varphi'^2, \tag{6}$$

$$\sigma'^2 = \frac{\varkappa}{3}\left[\frac{1}{2}\varphi'^2 + U(\varphi)\right]. \tag{7}$$

The field equation (2) for Friedman metric is the differential consequence of equations (6) and (7). Let us make a replacing

$$\sigma' = \Phi. \tag{8}$$

Then, the equation (6) can be written as:

$$\varphi' = -\frac{2}{\varkappa}\frac{d\Phi}{d\varphi} \tag{9}$$

and the equation (7) after replacing φ' by (9) takes a form

$$\frac{2}{3\varkappa}\left(\frac{d\Phi}{d\varphi}\right)^2 - \Phi^2 = -\frac{\varkappa}{3}U(\varphi). \tag{10}$$

The density of energy of the scalar field in that reference system $\rho = T_{44}$ is expressed by Φ

$$\rho = \frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 + U(\varphi) = \frac{2}{\varkappa^2}\left(\frac{d\Phi}{d\varphi}\right)^2 + U(\varphi) = \frac{3}{\varkappa}\Phi^2. \tag{11}$$

and is also positively defined.

The pressure p in given reference system looks like:

$$p = \frac{3}{\varkappa}\Phi^2 - 2U(\varphi). \tag{12}$$

Thus, the pressure, generally speaking, is not a function with a determined sign. Let us note, that while the derivation of the final forms of density and pressure of scalar field the equations (9) and (10) were used.

Further the set of exact solutions of equations (8-10) will be considered for the different types of nonlinearities.

2. Higgs non-linearity $U(\varphi) = \frac{\mu}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4$ $(\mu, \lambda < 0)$

In that case, the equation (10) admits the exact solution under an additional restriction on the coupling constant $(2\varkappa\mu = 4\lambda)$.

$$\Phi = \pm\frac{\varkappa}{4}\sqrt{-\mu}\,\varphi^2.$$

Choosing a sign plus, we have after integration (8) and (9) that $\varphi(T)$ and linear element ds^2 take a form

$$\varphi = \varphi(0)e^{-\sqrt{-\mu}T}, \tag{13}$$

$$ds^2 = -e^{-\frac{\varkappa}{4}\varphi^2(0)\exp(-2\sqrt{-\mu}T)}(dx^2 + dy^2 + dz^2) + dT^2. \tag{14}$$

The solution derived from (13) and (14) with a replacing T on $-T$ corresponds to the minus sign. For a sake of simplicity, let us set $\varphi(0) = \sqrt{\frac{\mu}{\lambda}} = \sqrt{\frac{4}{3\varkappa}}$. Then,

$$\varphi = \frac{2}{\sqrt{3\varkappa}} e^{-\sqrt{-\mu}T}; \qquad ds^2 = -e^{-\frac{1}{3}\exp(-2\sqrt{-\mu}T)}(dx^2 + dy^2 + dz^2) + dT^2. \tag{15}$$

Calculating ρ and p for the solution (15) we find

$$\rho = \frac{(-\mu)}{3\varkappa} e^{-4\sqrt{-\mu}T}, \quad p = \frac{(-\mu)}{3\varkappa} e^{-2\sqrt{-\mu}T}(4 - e^{-2\sqrt{-\mu}T}). \tag{16}$$

Analyzing formulae (16) for pressure and density under different asymptotes

a) $T \ll -\frac{\ln 2}{\sqrt{-\mu}}$, $p \approx \frac{\mu}{3\varkappa} e^{-4\sqrt{-\mu}T} = -\rho$; $T \to -\infty$, ρ, $p \to \infty$

with saving the equation of state $\rho = -p$, which is an equations of vacuum state. A scale factor in metric (15) tends to zero when $T \to -\infty$, and because of it the obtained solution corresponds to singularity in infinite past.

b) $T = -\frac{\ln 2}{\sqrt{-\mu}}$, $p = 0$, $\rho = \frac{(-\mu)16}{3\varkappa}$;
$T < -\frac{\ln 2}{\sqrt{-\mu}}$, $p < 0$;
$T > -\frac{\ln 2}{\sqrt{-\mu}}$, $p > 0$.

c) $T \gg -\frac{\ln 2}{\sqrt{-\mu}}$, $\rho = \dfrac{3\varkappa p^2}{16(-\mu)}$

In general case, excluding from the equations (16) T, we have the equation of state

$$p = \frac{(-\mu)}{3\varkappa}\sqrt{\frac{3\varkappa\rho}{(-\mu)}}\left(4 - \sqrt{\frac{3\varkappa\rho}{(-\mu)}}\right).$$

Let us note that when $T \to \infty$ the metric (15) transforms to Minkowski space metric.

3. Non-linearity of such type $U(\varphi) = \mu\cos 2\sqrt{\frac{3}{2}}\varkappa\varphi$, $(\mu > 0)$

Non-linearity of such type leads to the Sin-Gordon equation for the field function φ.

Equation (10) has the following exact solution:

$$\Phi = \sqrt{\frac{\varkappa}{3}}\mu \sin\left(\sqrt{\frac{3}{2}}\varkappa\varphi\right). \tag{17}$$

Potential energy $U(\varphi)$ reaches its extremum in the points $\sqrt{\frac{3}{2}}\varkappa\varphi = \frac{n\pi}{2}$. If n is even we obtain the minimum, if n odd, we obtain the maximum. Pressure and density look like:

$$\rho = \mu\sin^2\left(\sqrt{\frac{3}{2}}\varkappa\varphi\right); \qquad p = \mu\left[2 - 3\sin^2\sqrt{\frac{3}{2}}\varkappa\varphi\right]. \tag{18}$$

The equation of state looks like $p = 2\mu - 3\rho$. In the minimum point $U(\varphi)$ $\rho = 0$; $p = 2\mu$, in the maximum point $\rho = \mu$; $p = -\mu$, then, $p = -\rho$ in the maximum one $U(\varphi)$.

Integrating equation (9) we find

$$\frac{1}{2}\sqrt{\frac{3}{2}}\varkappa\varphi + \frac{\pi}{4} = \pm\arctan\left(Be^{-3\sqrt{\frac{\varkappa\mu}{3}}T_n^+}\right) + n\pi. \tag{19}$$

Requesting under relation $T_n^{\pm} = 0$ energy has the minimum we find B=1.

For the sake of definiteness let us choose sign plus in the formula 19. Then $T_n^{\pm} = 0$ changes from $-\infty$ to $+\infty$ for each n.

Integrating equation (8) we find

$$\sigma = \frac{1}{3}\ln\left|\cos\left(\sqrt{\frac{3}{2}}\varkappa\varphi\right)\right|. \tag{20}$$

With a chose a plus sign in (19) $\cos\left(\sqrt{\frac{3}{2}\varkappa\varphi}\right) > 0$ and using couple of elementary trigonometric transformations we obtain

$$\sigma = \frac{1}{3}\ln\left(\frac{2e^{-3\sqrt{\frac{\varkappa\mu}{3}}T_n^+}}{1+e^{-6\sqrt{\frac{\varkappa\mu}{3}}T_n^+}}\right). \tag{21}$$

A relevant linear element looks like:

$$ds^2 = -\left[\frac{2e^{-3\sqrt{\frac{\varkappa\mu}{3}}T_n^+}}{1+e^{-6\sqrt{\frac{\varkappa\mu}{3}}T_n^+}}\right]^{\frac{2}{3}}(dx^2+dy^2+dz^2)+dT^2.$$

Let us note, that because of (19) ρ and p are finite quantities with any T_n^+ and, consequently, space doesn't have physical singularities connected with equivalences to infinity of density and pressure. If $T_n^+ \to \infty$ then scale factor tends to zero $\sim e^{-2\sqrt{\frac{\varkappa\mu}{3}}T_n^+}$, and the equation of state becomes as a vacuum type $\rho = -p$. If $T_n^+ \to \infty$ then $e^{2\sigma} \to 0 \sim e^{-2\sqrt{\frac{\varkappa\mu}{3}}T_n^+}$ and the equation of state becomes as a vacuum type too.

Thus, we can think that the Universe is born if $T_n^+ = \infty$, according to the watch of observer in that Universe, from vacuum for characteristic time $T_{n\,char.}^+ \approx \left(2\sqrt{\frac{\varkappa\varphi}{3}}\right)^{-1}$, then it expends reaching the minimum of potential energy $U(\varphi)$ when the scale factor equals to 1; then the Universe shrinks and once again it absorbs by vacuum for the same characteristic time. And the process goes again and again for the infinity. Choice of the sign minus of arctangent in (19) gives the same result.

4. Massive linear scalar field $U(\varphi) = \dfrac{m^2c^2}{2\hbar^2} - U_0$

Before all let us note that the addition to the potential $U(\varphi)$ of additive constant U_0 does not change the scalar field equation but gives contribution in the energy-momentum tensor, which is equivalent to the insertion of the cosmological term. At the same time, the presence of this additive term makes it possible in some cases to obtain examples of exact solutions of the Einstein equation corresponding to a particular cosmological model. For the massive linear field with the potential of interaction $U(\varphi) = \frac{m^2c^c}{2\hbar^2}\varphi^2 - U_0$ equation (10) admits the exact solution if $U_0 = \frac{1}{3\varkappa}\frac{m^2c^2}{\hbar^2}$:

$$\Phi = \frac{mc}{\hbar}\sqrt{\frac{\varkappa}{6}}\varphi, \tag{22}$$

$$\varphi = -2\frac{2mc}{\hbar}\sqrt{\frac{1}{6\varkappa}}T, \qquad \sigma = -\frac{m^2c^2}{6\hbar^2}T^2, \tag{23}$$

$$\rho = \frac{1}{3\varkappa}\left(\frac{mc}{\hbar}\right)^4 T^2, \qquad p = \frac{2}{3\varkappa}\frac{m^2c^2}{\hbar^2} - \frac{1}{3\varkappa}\left(\frac{mc}{\hbar}\right)^4 T^2. \tag{24}$$

The equation of state looks like $p = \frac{2}{3\varkappa}\frac{m^2c^2}{\hbar^2} - \rho$. If $T = 0$ $U(\varphi)$ has a minimum, density of field energy $\rho = 0$; $p = \frac{2}{3\varkappa}\frac{m^2c^2}{\hbar^2}$. In cases of bigger T $\rho = -p \to \infty$.

5. Massive scalar field with cubic non-linearity $U(\varphi) = \dfrac{m^2c^2}{2\hbar^2}\varphi^2 - \dfrac{\lambda}{4}\varphi^4 - U_0$

The equation (10) admits the exact solution of type $\Phi = A\varphi^2 + B$, where A, B, U_0 are related by the following ratios.

$$A = \sqrt{-\frac{\varkappa\lambda}{12}}, \quad \lambda < 0, \quad B = \frac{\frac{\varkappa}{6}\cdot\frac{m^2c^2}{2\hbar^2 - \frac{2\lambda}{9}}}{2\sqrt{\frac{\varkappa\lambda}{12}}}, \quad U_0 = -\frac{3B^2}{\varkappa}. \tag{25}$$

After integration of equations (8) and (9) we find φ and linear element ds^2 in the form

$$\varphi = \varphi_0 e^{-\frac{4A}{\varkappa}T}, \tag{26}$$

$$ds^2 = -e^{-\frac{\varkappa\varphi_0^2}{4}\exp(-\frac{8AT}{\varkappa}) + B\varkappa T}(dx^2 + dy^2 + dz^2) + dT^2. \tag{27}$$

With the bigger T we have De Sitter world in nonstationary coordinate system. If $T \to \infty$ $\rho = -p$, i.e. also in this case, the moment of birth of the Universe corresponds to the equation of vacuum state, besides density and pressure in that case become in infinity, then the Universe expands and goes in De Sitter vacuum if $T \to +\infty$.

6. Non-linearity of type $U(\varphi) = \alpha e^{\beta\varphi}$

In that case the equation (10) has the exact solution

$$\Phi = \sqrt{\frac{\frac{\varkappa}{3}\alpha}{1 - \frac{\beta^2}{6\varkappa}}}e^{\frac{\beta}{2}\varphi},$$

where are eather $\alpha > 0$, $|\beta| < \sqrt{6\varkappa}$ or $\alpha < 0$, $|\beta| > \sqrt{6\varkappa}$.

Integrating equations (8) and (9) we have

$$\varphi = -\frac{2}{\beta}\ln\left(\sqrt{\frac{\frac{\varkappa}{3}\alpha}{1 - \frac{\beta^2}{6\varkappa}}}\frac{\beta^2}{2\varkappa}T\right),$$

$$\sigma = \frac{2\varkappa}{\beta^2}\ln\frac{T}{T_0}.$$

A relevant linear element looks like:

$$ds^2 = -\left(\frac{T}{T_0}\right)^{\frac{4\varkappa}{\beta^2}}(dx^2 + dy^2 + dz^2) + dT^2;$$

$$\rho = \frac{\alpha}{1 - \frac{\beta^2}{6\varkappa}}e^{\beta\varphi}; \qquad p = -\alpha e^{\beta\varphi}.$$

Thus, for exponential nonlinearity the equation of state is barotropic one and it looks like

$$p = \left(\frac{\beta^2}{6\varkappa} - 1\right)\rho.$$

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