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TO THE METHOD OF THE KINETIC EQUATION IN GENERAL RELATIVITY

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***Abstract*:** In this paper, we present a method for finding the 4-vector of the current and the energy momentum tensor of a nonequilibrium relativistic gas placed in a gravitational field, on the basis of the kinetic equation, which we write in a different, clearly covariant form, and an expression calculated using this method for 4- the current vector, which is valid for sufficiently weak and slowly varying fields (see (2.5)).

The designations adopted in this work, with the exception of those stipulated in the text, coincide with the designations adopted in the works [1] – [4].

***Keywords*:** relativistic kinetic theory, mathematical methods, tensore operators, partial covariant derivation, perturbative thepory.

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1. Explicitly covariant ways of writing the kinetic equation

The kinetic equation in general relativity is (see [1])

$$\hat{K} f_a(x,p) = \mathrm{I}_a(x,p), \tag{1.1}$$

where

$$\hat{K} = p^i \frac{\partial}{\partial x^i} - \Gamma^{\alpha}_{ik} p^i p^k \frac{\partial}{\partial p^{\alpha}},$$

can be written in an explicitly covariant form by introducing invariant momentum components

$$p^{(m)} = \xi^{(m)}_i p^i, \quad p^i = \eta^i_{(m)} p^{(m)}.$$

Here $\xi^{(m)}_i$ is an arbitrary quadruple of vectors linearly independent at each point, and $\eta^i_{(m)}$ is a quadruple with $\xi^{(m)}_i$ vectors ($\xi^{(m)}_i \eta^i_{(n)} = \delta^{(n)}_{(m)}$). Choosing $p^{(\alpha)}$ as variables in the equation (1.1), we reduce the operator $\hat{K}$ to the form

$$\hat{K} = p^{(m)} \eta^i_{(m)} \frac{\partial}{\partial x^i} + p^{(m)} p^{(n)} \eta^i_{(m)} \eta^j_{(n)} \xi^{(\alpha)}_{i;j} \frac{\partial}{\partial p^{\alpha}}, \tag{1.2}$$

where semicolon stands for covariant derivative. If the space admits several linearly independent Killing vectors at each point, then the operator $\hat{K}$ is simplified by choosing Killing vectors as $\xi^{(m)}_i$. If we choose an orthoreper as $\xi^{(m)}_i$, then we get an orthoreper notation of the kinetic equation (1.1).

Note one more property of the operator $\hat{K}$, which is verified by direct computation

$$\hat{K} a_{i_1 i_2 \ldots i_n} p^{i_1} p^{i_2} \cdots p^{i_n} = p^k p^{i_1} p^{i_2} \cdots p^{i_n} a_{i_1 i_2 \ldots i_n ;k}. \tag{1.3}$$

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This equality can be easily generalized if we bear in mind that the scalar function $f(x,p)$ can depend on the vector p^i only as a complex function $f(x,g(x,p))$, where $g(x,p)$ conventionally denotes the set of all scalars of the form $a_{i_1 i_2 \dots i_n} p^{i_1} p^{i_2} \cdots p^{i_n}$

$$\hat{K} f(x,p) = p^i \nabla_i f(x,p). \tag{1.4}$$

Here ∇_i means the derivative, which is calculated as ifp^i is a covariantly constant vector. We will call such a derivative the covariant partial derivative.[2]

Using (1.4), we can write (1.1) as

$$p^i \nabla_i f_a(x,p) = \mathrm{I}_a(x,p), \tag{1.5}$$

and the generalization of the kinetic equation to the case of electromagnetic forces [3] in the form

$$\left[p^i \nabla_i + \frac{e_a}{c} F^{\alpha}_{\cdot\, i} p^i \frac{\partial}{\partial p^\alpha}\right] f_a(x,p) = \mathrm{I}_a(x,p). \tag{1.6}$$

n what follows, we will need a linearized kinetic equation in the collisionless approximation for the case when the electromagnetic field is absent in the equilibrium state, and in the nonequilibrium state it is "on" weak electromagnetic and gravitational disturbance. As shown in [2,3], the distribution function in a nonequilibrium state should be sought in the form

$$f_a(x,p) = f_a^0(x,p)\{1 + \varphi_a(x,p)\}.$$

Then $\varphi_a(x,p)$ will satisfy the equation:

$$p^i \nabla_i \varphi_a(x,p) = \frac{e_a}{c} p^i \xi^k F_{ki} - p^i p^j \xi_k \delta \Gamma^k_{ij} \tag{1.7}$$

(in the corresponding equations obtained in [2,3], we replaced the operator $\hat{K}$ by $p^i \nabla_i$).

In (1.4) introduces the notion of a partial covariant derivative. Note one property of its

$$\left[\int dp p^{i_1} \cdots p^{i_n} J(x,p)\right]_{;j} = \int dp p^{i_1} \cdots p^{i_n} \nabla_j J(x,p). \tag{1.8}$$

Here $dp = \frac{\sqrt{-g} d^3 p}{p_4}$ is an invariant volume element in momentum space, $J(x,p)$ is a scalar or tensor function.

If we choose geodesic coordinates at the point Q as the coordinate system, then the equality (1.8) turns into a trivial statement about the possibility of differentiating an integral with respect to a parameter. Since the terms of this equality are components of the tensor, it takes place in any coordinate system and any point.

As an application of the property (1.8), we obtain the equations for the moments of the distribution function

$$f^{i_1 i_2 \dots i_n} = \int dp f(x,p) p^{i_1} \cdots p^{i_n}$$

by the following transformations of the kinetic equation:

$$\begin{gathered} p^i \nabla_i f_a(x,p) = \mathrm{I}_a(x,p), \\ \int dp p^{j_1} \cdots p^{j_{in}} p^i \nabla_i f_a)x,p) = \int dp p^{j_1} \cdots p^{j_{in}} \mathrm{I}_a(x,p), \\ \nabla_i f_a^{j_1 \dots j_n i}(x) = I_a^{j_1 \dots j_n}(x). \end{gathered} \tag{1.9}$$

As you can see, the concept of a partial covariant derivative is a rather convenient mathematical trick that allows you to avoid sometimes cumbersome calculations (compare, for example, the derivation of the equation (1.9) in [1]).

[2]At the time of writing this article (1975), the authors were unaware of the existing concept of the derivative in the Cartan fibration. In later works, this derivative is denoted $\tilde{\nabla}_i$ and is called the Cartan derivative.

2. Calculation of the 4-vector of the current of a nonequilibrium relativistic gas

The perturbation of the 4-vector of the current and the energy tensor — the momentum of a relativistic gas in a nonequilibrium state is determined by the formulas:

$$j^k = \sum_a e_a c \int dp p^k f_a^0(x,p)\varphi_a(x,p), \tag{2.1}$$

$$t^{ik} = \sum_a c \int dp p^i p^k f_a^0(x,p)\varphi_a(x,p). \tag{2.2}$$

These expressions are valid if $\delta\Gamma^k_{ij} = 0$. In general, $j^k = j^k_0 + j^k_1$, $t^{ik} = t^{ik}_0 + t^{ik}_1$, where j^k_1, t^{ik}_1 are calculated from (2.1) and (2.2), and j^k_0, t^{ik}_0 are easily found from the equilibrium distribution function (see [2]).

In the expressions (2.1), (2.2) it is necessary to substitute the solution of the equation (1.7) in the collisionless approximation. We will write this solution in the form

$$\varphi_a(x,p) \underset{\delta\to 0}{=} \int\limits_0^\infty d\tau \mathrm{e}^{-\delta\tau - \tau p^i\nabla_i}\left[\frac{e_a}{c}p^i\xi^k F_{ki} - p^i p^j \xi_k \delta\Gamma^k_{ij}\right]. \tag{2.3}$$

Substitution of (2.3) into (2.1) and (2.2) followed by formal integration over momenta (considering that ∇_i is a partial covariant derivative that does not act on p^i) is possible only if the commutators $[\xi_i, \nabla_j]$ and $[\nabla_i, \nabla_j]$ are equal to zero, that is, only in flat space. Obviously, in this case, when the unperturbed metric (background metric) admitting an equilibrium state of the gas is flat, the expressions obtained in this way will coincide after the Fourier transform with the corresponding expressions for a homogeneous plasma.

In the case when the background metric is not flat, i.e., the indicated commutators do not vanish, we will proceed as follows. Substitute (2.3), for example, into (2.1) and expand in a power series

$$j^k_a = e_a c \int\limits_0^\infty d\tau \mathrm{e}^{-\delta\tau} \sum_{n=0}^\infty \frac{(-\tau)^n}{n!} \int dp f_a^0(x,p) p^k p^{i_1}\dots p^{i_n}\nabla_{i_1}\dots\nabla_{i_n}\frac{e_a}{c}\xi_\ell F^\ell_{\cdot\, j} p^j. \tag{2.4}$$

After that, we count in some approximation each of the members of the series (2.4), and then again sum up the series. This can be done, for example, when the conditions

$$v_T \ll v_g \approx \lambda\omega_g, \tag{2.5a}$$

$$\lambda \ll L, \tag{2.5b}$$

where λ and L are the characteristic size of the disturbance and background inhomogeneity, respectively, and ω_g is equal in order of magnitude to $c\partial_i g_{k\ell}$. Obviously, v_g is the speed acquired by a particle in the gravitational field during the period of wave oscillation (see [4]). This gives the following expression for the 4-current vector (for simplicity, we put $\delta\Gamma^k_{ij} = 0$):

$$j^k_a = e_a^2 \int dp f_a^0(x,o)\left[\frac{p^k p^j \xi^\ell F_{\ell j}}{p^i\nabla_i} - \frac{p^k p^j p^m p^n \hat{T}_{mn}\xi^\ell F_{\ell j}}{(p^i\nabla_i)^3}\right], \tag{2.6}$$

where you need to integrate in a formal way, and the result of integration is written in the form

$$j^k_a = [\xi^k\xi^i\sigma_1(\hat{\xi},\nabla^2) + g^{ki}\sigma_2(\hat{\xi},\nabla^2) + \sigma_3(\hat{\xi},\nabla^2)\nabla^i\nabla^k + \xi^i\sigma_4(\hat{\xi},\nabla^2)\nabla^k + \xi^k\sigma_4(\hat{\xi},\nabla^2)\nabla^i]\xi^\ell F_{\ell i}.$$

Here $\hat{\xi} = \xi^i\nabla_i$, $\nabla^2 = g^{ij}\nabla_i\nabla_j$ are operators that can be considered commuting in our approximation, and $\sigma_i(\hat{\xi},\nabla^2)$ — scalar operator expressions. In (2.6), the tensor operator $\hat{T}_{mn}$ is introduced, which is defined by the relations

$$\nabla^i\hat{T}_{im} = 0, \qquad \xi^i\xi^j\hat{T}_{ij} = [\xi^i(\nabla_i\xi^j)\nabla_j] = \hat{\eta}. \tag{2.7}$$

The derivation of the formula (2.6) is given in Appendix I.

Appendix I. Derivation of formula (2.6)

We will start from the expression for the 4-current vector (2.4), which we rewrite in a different way, using the equality

$$\int dp f_a^0(x,p)p^{i_1}\cdots p^{i_n}=\sum_{\ell=0}^{n}(\xi,g)^{i_1\ldots i_\ell|i_{\ell+1}\ldots i_n}\sigma_\ell^{(n)}, \tag{I.1}$$

where

$$(\xi,g)^{i_1\ldots i_\ell|i_{\ell+1}\ldots i_n}=\xi^{(i_1}\cdots\xi^{i_\ell}g^{i_{\ell+1}i_{\ell+2}}\cdots g^{i_{n-1}i_n)}.$$

Substituting (I.1) in (2.4), we get

$$j_a^k=e_a^2\int\limits_0^\infty d\tau\sum_{n=0}^{\infty}\frac{(-\tau)^n}{n!}\sum_{\ell=0}^{n+2}\sigma_{n+2}^{(\ell)}(\xi,g)^{i_1\ldots i_\ell|i_{\ell+1}\ldots i_n kj}\nabla_{i_1}\cdots\nabla_{i_n}\xi^\ell F_{\ell j}. \tag{I.2}$$

Obviously, $\sigma_{n+2}^{(\ell)}=0$ for $n-\ell+2=2p+1$. Also the expression

$$(\xi,g)^{i_1\ldots i_\ell|i_{\ell+1}\ldots i_n kj}\nabla_{i_1}\cdots\nabla_{i_n}=\frac{\ell(\ell-1)}{(n+1)(n+2)}\xi^i\xi^k(\xi,g)^{i_1\ldots i_{\ell-2}|i_{\ell-1}\ldots i_n}\nabla_{i_1}\cdots\nabla_{i_n}$$

$$+\frac{n+2-\ell}{(n+1)(n+2)}g^{jk}(\xi,g)^{i_1\ldots i_\ell|i_{\ell+1}\ldots i_n}\nabla_{i_1}\cdots\nabla_{i_n}+\frac{(n+2-\ell)\ell}{(n+1)(n+2)}\xi^j g^{k(i_1}(\xi,g)^{i_1\ldots i_\ell|i_{\ell+1}\ldots i_n)}\nabla_{i_1}\cdots\nabla_{i_n}$$

$$+\frac{(n+2-\ell)\ell}{(n+1)(n+2)}\xi^k g^{j(i_1}(\xi,g)^{i_1\ldots i_\ell|i_{\ell+1}\ldots i_n)}\nabla_{i_1}\cdots\nabla_{i_n} \tag{I.3}$$

$$+\frac{(n+2-\ell)(n-\ell)}{(n+1)(n+2)}g^{k(i_1}g^{|j|(i_2}(\xi,g)^{i_3\ldots i_{\ell+2}|i_{\ell+3}\ldots i_n)}\nabla_{i_1}\cdots\nabla_{i_n}.$$

It is easy to get an approximate value for $\xi^{i_1}\cdots\xi^{i_n}\nabla_{i_1}\cdots\nabla_{i_n}$:

$$\xi^{i_1}\cdots\xi^{i_n}\nabla_{i_1}\cdots\nabla_{i_n}\approx\hat{\xi}^n-\frac{n(n-1)}{2}\hat{\eta}\;\hat{\xi}^{n-2} \tag{I.4}$$

(members containing $(\nabla_i\xi^k)^n$ и $(\nabla_{i_1}\cdots\nabla_{i_n}\xi^k)$ при $n\gg 2$, are discarded).

In the same approximation, the commutator $[\nabla_i,\nabla_k]$ in (I.3) should be set equal to zero. Therefore, (I.3), taking into account (I.4), takes the form

$$(\xi,g)^{i_1\ldots i_\ell|i_{\ell+1}\ldots i_n kj}\nabla_{i_1}\cdots\nabla_{i_n}=\frac{\ell(\ell-1)}{(n+1)(n+2)}\xi^i\xi^k\left[\hat{\xi}^{\ell-2}-\frac{(\ell-2)(\ell-3)}{2}\hat{\eta}\hat{\xi}^{\ell-4}\right]\nabla^{n-\ell+2}$$

$$+\frac{(n+2-\ell)\ell}{(n+1)(n+2)}g^{jk}\left[\hat{\xi}^{\ell}-\frac{\ell(\ell-1)}{2}\hat{\eta}\hat{\xi}^{\ell-2}\right]\nabla^{n-\ell}+\frac{(n+2-\ell)\ell}{(n+1)(n+2)}\xi^j\left[\hat{\xi}^{\ell-1}-\frac{(\ell-1)(\ell-2)}{2}\hat{\eta}\hat{\xi}^{\ell-3}\right]\nabla^{n-\ell}\nabla^k$$

$$+\frac{(n+2-\ell)\ell}{(n+1)(n+2)}\xi^k\left[\hat{\xi}^{\ell-1}-\frac{(\ell-1)(\ell-2)}{2}\hat{\eta}\hat{\xi}^{\ell-3}\right]\nabla^{n-\ell}\nabla^j$$

$$+\frac{(n+2-\ell)\ell}{(n+1)(n+2)}\left[\hat{\xi}^{\ell}-\frac{\ell(\ell-1)}{2}\hat{\eta}\hat{\xi}^{\ell-2}\right]\nabla^{n-\ell-2}\nabla^k\nabla^j.$$

This equality can be written as

$$(\xi,g)^{i_1\ldots i_\ell|i_{\ell+1}\ldots i_n jk}\nabla_{i_1}\cdots\nabla_{i_n}=\widetilde{(\xi,g)}^{i_1\ldots i_\ell|i_{\ell+1}\ldots i_n jk}\nabla_{i_1}\cdots\nabla_{i_n}$$

$$-\frac{n(n-1)}{2}\widetilde{(\xi,g)}^{i_1\ldots i_\ell|i_{\ell+1}\ldots i_n jk}\nabla_{i_1}\cdots\nabla_{i_{n-2}}\hat{T}_{i_{n-1}i_n}, \tag{I.5}$$

where $\widetilde{(\xi,g)}^{i_1\ldots i_\ell|i_{\ell+1}\ldots i_n jk}\nabla_{i_1}\cdots\nabla_{i_n}$ is formed from $(\xi,g)^{i_1\ldots i_\ell|i_{\ell+1}\ldots i_n jk}\nabla_{i_1}\cdots\nabla_{i_n}$ as follows: we consider ξ^i and ∇_k commuting until then after commutations we get an expression in which in the first place there are ∇_k, not convoluted with the vector ξ^i, and on the second, the third and fourth places are the corresponding operators is formed similarly $\hat{\xi}^p$, ∇^{2m} and ∇^j. Similarly, it is formed and

$\widetilde{(\xi,g)}^{i_1\ldots i_\ell|i_{\ll+1}\ldots i_n jk}\nabla i_1\cdots\nabla_{i_{n-2}}$ from $(\xi,g)^{i_1\ldots i_\ell|i_{\ell+1}\ldots i_n jk}\nabla_{i_1}\cdots\nabla_{i_{n-2}}$. Now it remains to substitute (I.5) in (2.4) and take into account (I.1)

$$k^k_a = e^2_a\int\limits_0^\infty d\tau \mathrm{e}^{-\delta\tau}\int dp f^0_a(x,p)\sum_{n=0}^\infty\frac{(-\tau)^n}{n!}(p^i\nabla_i)^n p^k p^j\xi^\ell F_{k\ell} \tag{I.6}$$

$$-\frac{1}{2}\sum_{n=0}^\infty\frac{(-\tau)^{n+2}}{n!}(p^i\nabla_i)^n p^\ell p^m p^k p^j \hat{T}_{\ell m}\xi^n F_{nj} = e^2_a\int dp f^0_a(x,o)\left[\frac{p^k p^j\xi^\ell F_{\ell j}}{p^i\nabla_i} - \frac{p^k p^j p^m p^n\hat{T}_{mn}\xi^\ell F_{\ell j}}{(p^i\nabla_i)^3}\right].$$

Estimates show that the above calculations are valid under conditions (2.5a)–(2.5b).

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