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О МЕТРИКЕ КЕРРА-НЬЮМЕНА *

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Подход с метрикой типа Керра-Шилда с использованием массы и электрического заряда уединенного астрофизического объекта как независимых параметров приводит к «выводу» решения Керра-Ньюмена для уравнений Эйнштейна-Максвелла.

Ключевые слова: уравнения Эйнштейна-Максвелла, точные статические решения уравнений Эйнштейна-Максвелла, решения Керра и Керра-Ньюмена.

ON KERR-NEWMAN METRIC

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The approach with the Kerr-Shild type metric with the use of the mass and electric charge of a solitary astrophysical object as independent parameters leads to the "derivation" of the Kerr-Newman solution for the Einstein-Maxwell equations.

Keywords: the Einstein-Maxwell equations, exact static solutions of the Einstein-Maxwell equations, solutions of Kerr and Kerr-Newman.

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Introduction

Well known Kerr-Newman solution was for the first time found out of the analogy considerations. The analysis of direct ways of its derivation can spotlight correlations with other solutions [1]. Shiffer with co-authors [2] worked out the original method which they used for a simple derivation of the Kerr solution [3]. We make use it here for finding the Kerr-Newman field and make a point of mentioning connection of complex harmonic function γ with electromagnetic part of the Kerr-Newman solution.

1. Kerr solution

At first let's enounce the essence of method offered by Shiffer with co-authors [2] to the extent in which it is necessary for us. We take the Kerr-Shild metric [4] as

$$g_{\mu\nu} = \delta_{\mu\nu} - 2Hl_\mu l_\nu; \qquad g = det(g_{\mu\nu}) = -1, \tag{1}$$

where $\delta_{\mu\nu} = diag(1, -1 - 1 - 1)$; l_μ is null vector having attributes

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$$l_\mu l^\mu = 0; \quad l_0 = 1; \quad l_{\mu,\nu} l^\mu = 0; \tag{2}$$

partial derivative is marked as a comma; greek indexes run through values $0, 1, 2, 3$; function H is proportional to any parameter m, which can be identified further with the mass of a system:

$$H = m \cdot f \tag{3}$$

(f is function to be found). As parameter m is any, then the vakuum Einstien's equations are split into systems of equations, corresponding to different exponential orders of parameter m. One from these systems of equations leads to condition

$$l_{\mu,\nu} l^\nu = 0. \tag{4}$$

For stationary vakuum field

$$f_{,n,n} = 0; \tag{5}$$

$$(fl^n)_{,n,k} + (fl_k)_{,n,n} = 0; \tag{6}$$

$$(fl_i l_k)_{,n,n} + (fl^n l_i)_{,n,k} + (fl_k l^n)_{,i,n} = 0, \tag{7}$$

where latin indexes run through values $1, 2, 3$. In the article [2] it is shown that from here taking into account (2), (4) only form of $l_{i,k}$, follows

$$l_{i,k} = -\alpha(\delta_{ik} + \beta \varepsilon_{ikp} l_p), \tag{8}$$

where α, β are scalar functions, $\delta_{ik} = -\delta^i_k$, and ε_{ikp} is 3D Livi-Chivita symbol. Functions α and β satisfy such relations that it is advisable to unite them into general harmonic complex function

$$\gamma = \alpha + i\beta, \tag{9}$$

satisfying two equations

$$\gamma_{,n,n} = 0; \tag{10}$$

$$\omega_{,n}\, \omega_{,n} = 1, \tag{11}$$

where

$$\omega = \frac{1}{\gamma} = \sqrt{x^2 + y^2 + (z - ia)^2} \tag{12}$$

and a is some constant parameter interpreted below as relative angular momentum of the system. Then we have

$$l_k = \frac{\omega_{,k} + \omega^*_{,k} - i\, \varepsilon_{kmp}\, \omega_{,m}\, \omega^*_{,p}}{1 + \omega_{,n}\, \omega^*_{,n}}, \tag{13}$$

where an asterisk marks complex conjugate, and

$$f = Re\, \gamma \equiv \alpha. \tag{14}$$

For ω from (12) we get known Kerr solution [3].

2. Kerr-Newman solution

In order to find the Kerr-Newman solution it is necessary to solve the Einstein equations in form

$$R_{\mu\nu} = -\varkappa T_{\mu\nu} \tag{15}$$

together with the Maxwell equations

$$(\sqrt{-g}\, F^{\mu\nu})_{,\nu} = 0; \qquad (\sqrt{-g}\, F_{*}^{\mu\nu})_{,\nu} = 0, \tag{16}$$

where $\varkappa = 8\pi$ when the light speed is $c = 1$ and the Newton gravitational constant is $G_N = 1$; $T_{\mu\nu}$ is the energy-momentum tensor of electromagnetic field

$$T_{\mu\nu} = -\frac{1}{8\pi}\,(F_{\mu\rho}\, F_{\nu}^{\rho} + F_{\mu\rho}^{*}\, F_{\nu}{}^{*\rho}), \tag{17}$$

$F^{\mu\nu} = -F^{\nu\mu}$ is the electromagnetic field tensor, and $F_{*}^{\mu\nu}$ is dually conjugate tensor to it. Function H is taken as

$$H = m \cdot f + q^2 h, \tag{18}$$

where f, h are functions which must be found, and m, q are arbitrary independent parameters interpreted below as a mass and an electric charge of the system.

Taking into account that for static solution of Reissner-Nordström

$$F_{0l} = -q\frac{x_l}{r^2} = q\mathcal{F}_{0l}, \tag{19}$$

we shall consider for us the interesting stationary case

$$F_{\mu\nu} = q\mathcal{F}_{\mu\nu}, \tag{20}$$

where

$$\mathcal{F}_{\mu\nu} = \alpha_{\nu,\mu} - \alpha_{\nu,\mu}\,; \quad \alpha_0 = \varphi. \tag{21}$$

Equations (13), (14) “split” with respect to degrees of parameters m, q give at the same time with old equations (5)-(7) additional equations in the form

$$h_{,n,n} = \varphi_{,n}\varphi_{,n} - \frac{1}{2}\mathcal{F}_{mn}\mathcal{F}_{mn}; \tag{22}$$

$$(hl^n)_{,n,k} + (hl_k)_{,n,n} = 2\varphi_{,n}\mathcal{F}_{kn}; \tag{23}$$

$$(hl^n l_i)_{,n,k} + (hl_i l_k)_{,n,n} + (hl_k l^n)_{,n,i} = \delta_{ik}(\varphi_{,n}\varphi_{,n} - \frac{1}{2}\mathcal{F}_{mn}\mathcal{F}_{mn}) + 2\varphi_{,i}\varphi_{,k} - 2\mathcal{F}_{in}\mathcal{F}_{in}; \tag{24}$$

$$\varphi_{,n,n} = 0; \tag{25}$$

$$\mathcal{F}_{ik,k} = 0; \tag{26}$$

$$(fl^m l^n)_{,m,n} = 0. \tag{27}$$

Let’s introduce quantities

$$\mathcal{E}_k = \varphi_{,k} \tag{28}$$

and

$$\mathcal{H}_k = \varepsilon_{knm}\alpha_{n,m} = -\frac{1}{2}\mathcal{F}_{nm}. \qquad (29)$$

In accordance with equation (25) we have

$$\varepsilon_{ikn}\mathcal{H}_{n,k} = 0, \qquad (30)$$

and taking into acount the expression (27) we get

$$\mathcal{H}_{k,k} = 0. \qquad (31)$$

Hence it is possible to introduce some harmonic function η such as

$$\mathcal{H}_k = -\eta_{,k}\,, \qquad (32)$$

where a negative sign is taken by analogy with (26). This at once allow naturally to introduce a harmonic complex vector

$$\mathcal{F}_k = \mathcal{E}_k + i\mathcal{H}_k = -\zeta_{,k}\,, \qquad (33)$$

where

$$\zeta = \varphi + i\eta \qquad (34)$$

is complex harmonic function if to take account (23) and (29).

It is worth mentioning that dual conjugation of $F_{\mu\nu}$ corresponds to a rotation in complex plane. The condition of equality to zero of the expression L_k may now be writtten as

$$\varphi_{,k} = l_k(\varphi_{,m}l_m) - \varepsilon_{kmn}\eta_{,n}l_m. \qquad (35)$$

Multiplying this expression by $l_i\,\varepsilon_{ijk}$, we get a similar expression for the gradient of function η in the form

$$\eta_{,k} = l_k(\eta_{,m}l_m) + \varepsilon_{kmn}\varphi_{,n}l_m. \qquad (36)$$

These two expressions can now be united if to take account (32),

$$\zeta_{,k} = l_k(\zeta_{,m}l_m) - i\varepsilon_{kmn}\zeta_{,n}l_m, \qquad (37)$$

then

$$\zeta_{,k}\zeta_{,k} = (l_k\zeta_k)^2. \qquad (38)$$

Because function ζ is a harmonic function, then it may be choosen as

$$\zeta = \frac{1}{\omega}, \qquad (39)$$

where ω is defined by expression (12). In this case we get the equations coinciding with (10), (11). Such choice of function ζ allows according to (31) to find the expression

$$\mathcal{F}_k = -\left(\frac{1}{\omega}\right)_{,k}, \qquad (40)$$

that is already known for the Kerr-Newman field [5].

An asymptotic form of expression (40) gives evidence that the electric and magnetic fields finded thus (see (31) correspond to the charged rotating source (q is a charge, ma is an angular momentum).

Let's pass on to finding the function h from the equations (20)-(22) which are written down in form

$$h_{,n,n} = -\mathcal{F}_n \mathcal{F}_n^*; \tag{41}$$

$$\varepsilon_{kin}\, \varepsilon_{ipm} (h l_p)_{m,n} = i \varepsilon_{knm} \mathcal{F}_n \mathcal{F}_m^*; \tag{42}$$

$$(hl^n)_{,n}(l_{i,k} + l_{k,i}) + 2h l_{k,n} l_{i,n} = \mathcal{F}_i^* \mathcal{F}_k + \mathcal{F}_i \mathcal{F}_k^* - i \mathcal{F}_n \mathcal{F}_p^* (\varepsilon_{knp} l_i + \varepsilon_{inp} l_k + (\delta_{ik} - l_i l_k) \mathcal{F}_n \mathcal{F}_n^*. \tag{43}$$

Taking into account that function $1/\omega$ is a harmonic function then equation (41) can be rewritten as

$$\left(h + \frac{1}{2} \cdot \frac{1}{\omega \omega^*} \right)_{,n,n} = 0. \tag{44}$$

From here we obtain, using the boundary conditions on infinity for elimination of an arbitrary constant of integration,

$$h = -\frac{1}{2} \cdot \frac{1}{\omega \omega^*} + \chi, \tag{45}$$

where χ is some harmonic function. Taking $i = k$ in (43), and using an explicit view $\mathcal{F}_n$, l_n, $l_{i,k}$ and also expression (37) with taking into account (31) we will obtain

$$h(\varphi_{,k} l_k) - \varphi(h_{,k} l_k) = -\frac{1}{2} \cdot \frac{1}{\omega \omega^*}, \tag{46}$$

So we obtain from (45) as the result $\chi \propto \varphi$. Therefore we have

$$h = -\frac{1}{2} \cdot \frac{1}{\omega \omega^*} + b\, \varphi. \tag{47}$$

Further we take b equal to constant. Now we can write the function H, which was introduced in (16), taking into account that $\varphi = f = Re\gamma$:

$$H = (m + b\, q^2) f - q^2 \frac{1}{2} \cdot \frac{1}{\omega \omega^*}. \tag{48}$$

In view of arbitrariness of parameters m and q we can exclude function χ, redefining these parameters. Then we have

$$h = -\frac{1}{2} \cdot \frac{1}{\omega \omega^*}, \tag{49}$$

and

$$-2H = -m \left(\frac{1}{\omega} + \frac{1}{\omega^*} \right) + q^2 \frac{1}{\omega \omega^*}. \tag{50}$$

Thus the finding of Kerr-Newman's solution is completed.

3. Conclusion

In present article the approach of Schiffer et el. [2] is used for "finding" of Kerr-Newman solution of the Einstein-Maxwell equations with metric in the Kerr-Schild form. For both the Kerr part and the electromagnetic part of the Kerr-Newman solution there are harmonic functions as solutions of 3D Laplace's equation. We obtain these functions by spliting the Einstein-Maxwell equations with respect to the degrees of mass and electric charge parameters. It is the harmonic functions that are at the heart of the Kerr-Newman metric.

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The final comments to this article.

The article of Finkelstein R.G. (The general relativistic fields of a charged rotating source. *Journal of Mathematical Physics, 1975, vol.16, no.6, p.1271) with analogical results was published later (June 1975) and without reference to the present article published 14 March 1975.*

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