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ВНУТРЕННЕЕ СФЕРИЧЕСКИ СИММЕТРИЧНОЕ СТАТИЧЕСКОЕ РЕШЕНИЕ УРАВНЕНИЙ ЭЙНШТЕЙНА-МАКСВЕЛЛА *

Баранов А. М.[a,1]

[a] ФГБОУ ВПО Красноярский государственный педагогический университет им.В.П.Астафьева (КГПУ), г. Красноярск, 660049, Россия

Получено точное внутреннее статическое решение уравнений Эйнштейна-Максвелла в координатах Бонди для электрически заряженного жидкого шара как источника решения Райснера-Нордстрема.

Ключевые слова: уравнения Эйнштейна-Максвелла, точные статические решения уравнений Эйнштейна-Максвелла, решение Райснера-Нордстрема, внутренний источник решения Райснера-Нордстрема.

AN INTERIOR SPERICAL STATIC SOLUTION OF EINSTEIN-MAXWELL EQUATIONS

Baranov A. M.[a,1]

[a] Krasnoyarsk State Pedagogical University named after V.P.Astafyev, 60049, Krasnoyarsk, Russia

The exact interior static solution of the Einstein-Maxwell equations in Bondi's coordinates for the electrically charged fluid ball is obtained as a source of the Reissner-Nordström solution.

Keywords: Einstein-Maxwell equations, static spherical exact solutions of the Einstein-Maxwell equations, Reissner-Nordström's solution, the interior Reissner-Nordström's source.

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Introduction

The Reissner-Nordström solution is well-known as an exterior solution of the Einstein-Maxwell equations with an electric charge [1]. Therefore it would be naturally to find an interior solution of a charged substance as an analog of Schwarzschild's interior solution [2].

Brahmachary [3, 4] obtained an interior solution as a combination of both solutions for a charged core without the substance and an uncharged spherical distribution of the matter around the core. Such description is not physically satisfactory. Therefore in present paper an attempt of finding of a static solution for the Einstein-Maxwell equations with a spherical distribution of a charged perfect fluid was done.

1. Method of τ-field and Einstein-Maxwell equations

The method of τ-field for an introduction of physically observable values is used here after [5, 6]. A physically observable strength of an electrical field and a charge density respectively can be written according to this formalism in a comoving frame of reference as

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[1]E-mail: alex_m_bar@mail.ru

$$E_\nu = \frac{H_{0\nu}}{\sqrt{g_{00}}}; \tag{1}$$

$$\rho = j^0 \sqrt{g_{00}}, \tag{2}$$

where $H_{\mu\nu}$ is a tensor of an electromagnetic field; g_{00} is a 00-component of a metric tensor; j^0 is a 0-component of an electrical 4-current density; (the greek subscripts run through the values 0, 1, 2, 3).

Further we shall write a square of the time-space interval using Bondi's coordinates as

$$ds^2 = F du^2 + 2D du dr - r^2 (d\theta^2 + sin^2\theta \, d\varphi^2), \tag{3}$$

where $F = g_{00}$; $D = g_{01}$, u is a retarded time coordinate; r is a radial coordinate and φ, θ are angle variables. The light velocity is here equal to an unit.

The Maxwell equations [7],

$$H^{\mu\nu}{}_{;\,\nu} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} H^{\mu\nu}\right)_{,\,\nu} = -4\pi j^\mu, \tag{4}$$

for spherical and static conditions can be rewritten in a form

$$\left(\frac{r^2 H_{01}}{D}\right)_{,\,r} = 4\pi r^2 \rho D F^{-1/2}, \tag{5}$$

where g is a metric tensor determinant; semicolon marks a covariant derivative; comma marks a private derivative.

The substance energy-momentum tensor without a viscosity is chosen in form

$$(T_{\mu\nu})_{matter} = (p + \mu) u^\mu u^\nu - p \, g_{\mu\nu}, \tag{6}$$

where p is a pressure; μ is a mass density; $u^\mu = \dfrac{dx^\mu}{ds}$.

To obtain a solution in terms of elementary functions we choose a function of mass density as

$$\mu(r) = \beta - \alpha \, r^2, \tag{7}$$

where β and α are constants.

For a physically observable density of an electrical field's energy in a charged substance a following functional dependence is supposed

$$W_{el} = -\frac{1}{8\pi} g^{\nu\mu} E_\nu E_\mu = -\frac{1}{8\pi} H_{0\nu} H_{0\mu} = \frac{1}{8\pi} \lambda r^2, \tag{8}$$

where $\lambda = const > 0$.

Now we shall rewrite the Einstein-Maxwell system of equations as

$$\frac{F}{rD} (lnD)_{,\,r} = \frac{\varkappa}{2} (p + \mu); \tag{9}$$

$$\frac{F}{rD^2} (lnD)_{,\,r} - \frac{1}{2D^2} (F_{,\,r,\,r} + \frac{2}{r} F_{,\,r} - F_{,\,r} (lnD)_{,\,r}) = -\varkappa \left(p + \frac{1}{8\pi} \frac{E^2}{D^2}\right); \tag{10}$$

$$\frac{1}{r^2 D^2} (-D^2 + F + rF_{,\,r} - rF (lnD)_{,\,r}) = -\varkappa \left(\frac{1}{2}(\mu - p) + \frac{1}{8\pi} \frac{E^2}{D^2}\right); \tag{11}$$

$$\left(\frac{r^2 F}{D}\right)_{,\,r} = 4\pi r^2 \rho D F^{-\frac{1}{2}}; \tag{12}$$

where $\varkappa = 8\pi$; Newton's gravitational constant γ is here equal to unit; $E = H_{0\,1}$ is a coordinate strength of the electric field.

2. Solution of Einstein-Maxwell equations

Now we take

$$D(r) = \left(F(r)\,\varepsilon(r)^{-1}\right)^{\frac{1}{2}}, \tag{13}$$

where $\varepsilon(r) = 1 + \varphi(r)$, $\varphi(r)$ is an unknown function for now.

The formula (13) is suggested by the known transformation from coordinates of curvatures to Bondi's coordinates for Schwarzschild's interior solution.

The junction conditions with an exterior solution of Reissner-Nordsröm can be written as

$$F(r) = 1 - \frac{2m}{a} + \frac{q^2}{a^2}; \quad F(r)_{,\,r} = \frac{2m}{a^2}(m - \frac{q^2}{a}); \tag{14}$$

$$W_{el} = \frac{q^2}{8\pi a^4}; \tag{15}$$

$$p(a) = 0; \quad D(a) = 1; \quad D_{,\,r}(a) = 0, \tag{16}$$

where q, m are the integral electric charge and mass of the ball respectively; a is a radius of ball.

In general the condition $D_{,\,r}(a) = 0$ can be not correct in chosen coordinates [2, 8].

Now we go over to equations (9)-(12). If we use equations (13), (8) and (7), we obtain from (11) new equation. A quadrature of this equation is equal to

$$\varphi(r) = -\eta r^2 + \xi r^4, \tag{17}$$

where $\eta = \dfrac{1}{3}\varkappa\beta$ and $\xi = \dfrac{\varkappa}{5}(\alpha - \dfrac{\lambda^2}{8\pi})$ are parameters.

Further we subtract the equation (10) from (9) and next subtract (11). Taking into account the equation (9), we get equation

$$\frac{F_{,\,r,\,r}}{2F}\varepsilon + \frac{F_{,\,r}}{F}\left(\frac{\varepsilon_{,\,r}}{2} - \frac{\varepsilon}{r}\right) - \frac{1}{4}\left(\frac{F_{,\,r}}{F}\right)^2 \varepsilon + \frac{1}{r^2} - \frac{\varepsilon}{r^2} + \frac{\varepsilon_{,\,r}}{2r} = \frac{\varkappa}{8\pi}\lambda^2 r^2. \tag{18}$$

Next we make substitution

$$U(r) = (lnF)_{,\,r}\,\varepsilon^{\frac{1}{2}}\,r^{-1}. \tag{19}$$

Finally, taking into account Eq.(17), we obtain Riccati's general equation [9]

$$U_{,\,r} + \frac{1}{2}r\varepsilon^{-\frac{1}{2}}\,U^2 = 2\nu\, r\,\varepsilon^{-\frac{1}{2}}, \tag{20}$$

where a parameter $\nu = \dfrac{\varkappa}{4\pi}\lambda^2 - \xi$.

The solution to the equation (20) strongly depends on variability of signs of ν and ξ, i.e. from a relation between parameters α and λ. Here we will consider an opportunity only with

$$0 < \alpha < \frac{\lambda^2}{8\pi}, \quad \text{then} \quad -\xi < 0; \;\; \nu > 0. \tag{21}$$

In this case the separation of variables may be made in the equation (20) and after the integration we have function

$$U(r) = 2\sqrt{\nu}\,\frac{R(a) + \dfrac{P}{Q}R(r)}{R(a) - \dfrac{P}{Q}R(r)}, \tag{22}$$

where

$$P = \eta - \varkappa\,\alpha\,a^2 - 4\xi\,a^2 - 2\sqrt{\nu\,\varepsilon(a)};$$

$$Q = \eta - \varkappa\,\alpha\,a^2 - 4\xi\,a^2 + 2\sqrt{\nu\,\varepsilon(a)};$$

$$R(r) = exp\Bigg(-\sqrt{\frac{\nu}{\xi}}\ \arcsin\left(\frac{\xi\,r^2 + \eta/2}{\sqrt{\xi + \eta^2/4}}\right)\Bigg).$$

The constant of integration is found here for the true junction condition of pressure (16). After that from (19) and (22)) we find

$$F(r) = \varepsilon(a)\,R(a)R(r)^{-1}\left(1 - \frac{P}{Q}\frac{R(r)}{R(a)}\right)^2\left(1 - \frac{P}{Q}\right)^{-2}. \tag{23}$$

The condition $D(a) = 1$ was here used to find the integration constant.

$$D(r) = (\varepsilon(a)\,R(a))^{1/2}\,(\varepsilon(r)R(r))^{-1/2}\left(1 - \frac{P}{Q}\frac{R(r)}{R(a)}\right)\left(1 - \frac{P}{Q}\right)^{-1}. \tag{24}$$

We find the pressure from ((9) as

$$p(r) = -\frac{1}{3}\beta + \alpha r^2 + 4\frac{\xi}{\varkappa}r^2 + \frac{2}{\varkappa}(\nu\varepsilon(r))^{1/2}\frac{\left(1 - \frac{P}{Q}\frac{R(r)}{R(a)}\right)}{\left(1 - \frac{P}{Q}\right)}. \tag{25}$$

This solution (see Eqs. (24), (25), (23) goes over into the standard solution of Schwarzschild for the perfect fluid if at first we shall take limit $\alpha \to 0$, and then go to the limit procedure $q \to 0$ with $\beta = \mu_0 = const$. As result we shall have (see e.g. [2])

$$\begin{aligned}
\lim_{q\to 0}\lim_{\alpha\to 0} F(r) &= \left(\frac{3\sqrt{1-\eta a^2} - \sqrt{1-\eta r^2}}{2}\right)^2;\\
\lim_{q\to 0}\lim_{\alpha\to 0} D(r) &= \frac{1}{2}\left(\frac{3\sqrt{1-\eta a^2}}{\sqrt{1-\eta r^2}} - 1\right);\\
\lim_{q\to 0}\lim_{\alpha\to 0} p(r) &= \mu_0\left(\frac{\sqrt{1-\eta r^2} - \sqrt{1-\eta a^2}}{3\sqrt{1-\eta a^2} - \sqrt{1-\eta r^2}}\right),
\end{aligned} \tag{26}$$

where $\eta = \dfrac{1}{3}\varkappa\mu_0$.

We will take $\mu(a) = 0$ to find α and β out of Eq.(14) in the form

$$\alpha = \frac{15}{\varkappa a^3}\left(m - \frac{3q^2}{5a}\right). \tag{27}$$

The condition $D_{,r}(a) = 0$ is here the true.

Further we find a restriction on an integral mass (in dimensional units), using (21) and $\lambda^2 = \dfrac{q^2}{a^6}$ out of (15),

$$\frac{3}{5}\frac{q^2}{ac^2} < m < \frac{2}{3}\frac{q^2}{ac^2}. \tag{28}$$

We have physically observable density of charge after using (12), (13), (8),

$$\rho(r) = \frac{q}{\left(\dfrac{4}{3}\pi a^3\right)} \cdot \varepsilon^{1/2}(r). \tag{29}$$

The function $\rho(r)$ is monotonic in range $0 \leq r \leq a$ and gets maximum at $r = 0$.

We must note, that at $r = 0$ the mass density must be positive. Then simple demand follows from (28) and (7):

$$m > \frac{1}{2}\frac{q^2}{ac^2}. \tag{30}$$

This demand is satisfied for the expression (29).

Herewith the observable strength of the electrical field equals to (in view of (1), (8)

$$E_1(r) = \frac{4}{3}\pi\rho r. \tag{31}$$

We see that $E_1(r)$ is simple generalization of an expression for electrical field strength of homogeneously charged ball in a "flat"case, i.e. for case with the zero gravitational constant.

The monotonicity demand of the function $E_1(r)$ inside of ball leads to lower limit of the ball radius

$$a > \frac{m\gamma}{c^2} = \frac{1}{2}R_g, \tag{32}$$

where R_g is the Schawzcshild radius.

Conclusion

In present work the spheric internal static solution of the Einstein-Maxwell equations is obtained. It can be considered as a source of the Reissner-Nordström external solution. The metric is taken in Bondi's form. The energy-momentum tensor (EMT) inside of the charged ball is taken as a superposition of EMT of the perfect fluid with the decreasing parabolic density function of mass and EMT of the electromagnetic field with parabolic density function of the electrical field's energy. The requirement of to lower limit of the ball radius is finded. The obtaind solution goes over into Schwarzschild's internal solution when the electric charge disappears.

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Авторы

Баранов Александр Михайлович, д.ф.-м.н., профессор, кафедра физики и методики обучения физике, ФГБОУ ВПО Красноярский государственный педагогический университет им.В.П.Астафьева (КГПУ), ул. Ады Лебедевой, 89, г. Красноярск, 660049, Россия
E-mail: alex_m_bar@mail.ru

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Authors

Baranov Alexandre Mikhailovich, Doctor of Sci., Professor, Krasnoyarsk State Pedagogical University named after V.P.Astafyev, 89 Ada Lebedeva St., Krasnoyarsk, 660049, Russia
E-mail: alex_m_bar@mail.ru

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