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ТИП ПОЛЯ И ТИП ТЕНЗОРА ЭНЕРИИ-ИМПУЛЬСА (НА АНГЛИЙСКОМ) *

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FIELD TYPE AND ENERGY-MOMENTUM TENSOR TYPE (IN ENGLISH)

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Let us consider a problem that naturally arises in general relativity and is not based on any hypotheses other than the assumption that the Einstein field equations hold

$$R_{\alpha\beta} - \frac{R}{2} q_{\alpha\beta} = \dot{T}_{\alpha\beta}. \tag{1}$$

As shown by the author [1, §20] in the most general case, for any possible distribution, motion, and properties of the matter creating the field (formally – for any energy-momentum tensor $T_{\alpha\ beta}$, which is limited only by the symmetry condition: $T_{[\alpha\beta]} = 0$) we can say that there are three and only three types of gravitation fields, if this kind of classification is made based on the consideration of the algebraic structure of a certain tensor (space tensor-matter) and, as a concrete implementation of this theorem, it is possible to define unambiguously at each point of space- time such an orthreper with respect to which the components of this tensor will have a well-defined form for each of the three possible types of gravitational fields.

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On the other hand, in our general formulation of the question, when the structure of the energy-momentum tensor $T_{\alpha\beta}$ is not predetermined in any way, it is natural to raise the question of the classification of gravitational fields depending on the algebraic structure of $T_{\alpha\beta}$.

It is obvious that (1) both of these classifications are fully defined for each specific case; (2) they are fully defined in the general case, in the sense of invariant definition of every possible sub-case. In view of this, any problem in the general theory of relativity, if it is correctly formulated, should be solved within the framework of these two classifications. In this connection, a problem arises: by classifying in detail the possible types of gravitational fields based on the logical intersection of the two classifications mentioned above, we determine each possible case in an invariant way.

This problem, which is important for field theory, is far from trivial. A complete solution to this problem will require considerable effort and time. In this paper, the correct formulation of the question is formulated; some basic facts that arise when solving it are obtained, and a specific case of electromagnetic radiation is considered as an illustration..

§1. Basic formulas for classifying gravitational fields

Let us first indicate the basic formulas for classifying gravitation fields of the most general form, based on the consideration of the space-time tensor. Consider the tensor construction

$$P_{\alpha\beta\gamma\delta} = \nu R_{\alpha\beta\gamma\delta} + \sigma S_{\alpha\beta\gamma\delta} + \rho g_{\alpha[\gamma} q_{\delta]\beta}, \tag{2}$$

where $R_{\alpha\beta\gamma\delta}$ is the space-time curvature tensor $\nu,\ \sigma,\ \rho$ are arbitrary invariants, and

$$S_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + g_{\alpha[\gamma} q_{\delta]\beta}, \tag{3}$$

We can then show that for any $\nu,\ \sigma,\ \rho$, there are three and only three types of algebraic structures of the tensor $R_{\alpha\beta\gamma\delta}$ that determine the classification of fields. The choice of $\nu,\ \sigma,\ \rho$ scalars is not predetermined yet, there is no need to specify one or the other of these scalars (except, of course, for the condition $\nu,\ \sigma \neq 0$.) We believe that we should proceed from the requirements of simplicity of some geometric considerations. The following selection of these scalars is particularly convenient:

$$\nu = 1, \quad \sigma = -1, \quad \rho = \frac{2}{3}T, \quad T = T^{\sigma}_{\sigma},$$

when $P_{\alpha\beta\gamma\delta}$ coincides with the space-time Weyl conformal curvature tensor $C_{\alpha\beta\gamma\delta}$. For the proof, it is sufficient, using the field equations (1), to express $T_{\alpha\beta}$ and T by $R_{\alpha\beta}$ and R (taking into account, in order to avoid confusion in the signs, that $R_{\alpha\beta} \overset{\text{def}}{=} R^{\sigma}_{\alpha\sigma\beta}$). We will focus further on this choice $\nu,\ \rho,\ \sigma$ and have [2, §28].

$$P_{\alpha\beta\gamma\delta} \equiv C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - S_{\alpha\beta\gamma\delta} + \frac{T}{3} g_{\alpha[\gamma} q_{\delta]\beta}.$$

Then we introduce the collective indices of the 6-dimensional bivector space[1, §15, §18], and write the conformal curvature tensor and the tensor $g_{\alpha[\gamma} q_{\delta]\beta}$ as two symmetric tensors C_{ab} and g_{ab} ($a,\ b = 1, \ldots, 6$). The numbering of skew-symmetric pairs is chosen as follows: 14–1, 24–2, 34–3, 23–4, 31–5, 12–6. Let us now consider pairs of tensors C_{ab} and $g_{ab}|g_{ab}| \neq 0$ (as can be easily seen by calculating it at the point), in the orthreper and corresponding to them λ-matrix $C_{ab} - \lambda g_{aq}$. We can formulate the basic one in the sense of the algebraic structure of the conformal curvature tensor. textitThere are three types of gravity fields, and these types are completely determined by the characteristic of the λ matrix:

$$\text{type I } [\text{III}, \overline{\text{III}}]; \quad \text{II type } [21, \overline{21}]; \quad \text{III type } [3, \overline{3}];$$

The numbers in parentheses denote exponents of elementary divisors λ-matrices; super-crossed numbers – exponents of elementary divisors with complex-conjugate bases corresponding to non-crossed numbers; for the third type, the bases are real, therefore, all equal to each other. For each tpe at a given point in

space-time, one can always choose (and unambiguously) an orthreper in such a way that the components of the Weyl conformal curvature tensor will be written in terms of collective indices.

$$(C_{ab}) = \left(\begin{array}{cc} [J|J]M & N \\ N & -M \end{array}\right) \quad (g_{ab}) = \left(\begin{array}{cc} [J|J]\varepsilon & 0 \\ 0 & -\varepsilon \end{array}\right), \tag{4}$$

where ε is an identity three-dimensional matrix, M, N are three-dimensional symmetric matrices defined as follows, respectively:

$$\text{Type I} \quad M = \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{pmatrix}, \quad N = \begin{pmatrix} \beta_1 & & \\ & \beta_2 & \\ & & \beta_3 \end{pmatrix} \sum_{i=1}^{3} \alpha_i = \sum_{i=1}^{3} \beta_i = 0.$$

$$\text{Type II} \quad M = \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 + 1 & \\ & & \alpha_2 - 1 \end{pmatrix}, \quad N = \begin{pmatrix} \beta_1 & & \\ & \beta_2 & 1 \\ & 1 & \beta_3 \end{pmatrix} \begin{array}{l} \alpha_1 + 2\alpha_2 = 0 \\ \beta_1 + 2\beta_2 = 0 \end{array} \tag{5}$$

$$\text{Type III} \quad M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ & & \beta_3 \end{pmatrix}.$$

If the case of empty space-time ($R_{\alpha\beta} = 0$) or the case of a homogeneous distribution of matter ($R_{\alpha\beta} = \varkappa g_{\alpha\beta}$), both cases are usually called Einstein spaces in geometry — then $C_{ab} = R_{ab} + \sigma g_{ab}$. This results in a canonical form for the components of the curvature tensor in the canonical orthreper [1, §20].

In general, it is also possible to obtain a canonical form for the curvature tensor in a canonical orthreperon, but up to the possible structure of the tensor $T_{\alpha\ \ beta}$. The corresponding formulas for $R_{\alpha\ \ beta\gamma\sigma}$ (R_{ab} – in collective indices) are obtained directly from (3) and (6) and will have the form, as you can easily see, in the orthreper where the formulas take place (6)

$$(R_{ab}) = \left(\begin{array}{cc} [J|J]M & N \\ N^* & P \end{array}\right),$$

где

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$$M = \begin{vmatrix} \alpha_1 + \frac{1}{2}(T_{11} - T_{44}) + \frac{1}{3}T, & \frac{1}{2}T_{12}, & \frac{1}{2}T_{13} \\ \frac{1}{2}T_{12}, & \alpha_2 + \frac{1}{2}(T_{22} - T_{44}) + \frac{1}{3}T, & \frac{1}{2}T_{23} \\ \frac{1}{2}T_{13,} & \frac{1}{2}T_{23}, & \alpha_3 + \frac{1}{2}(T_{33} - T_{44}) + \frac{1}{3}T. \end{vmatrix}$$

$$N = \left|\begin{array}{ccc} \beta_1 & \frac{1}{2}T_{34} & -\frac{1}{2}T_{24} \\ -\frac{1}{2}T_{34} & \beta_2 & \frac{1}{2}T_{14} \\ \frac{1}{2}T_{24} & -\frac{1}{2}T_{14} & \beta_3 \end{array}\right| \begin{array}{l} \sum \alpha_i = 0 \\ \sum \beta_i = 0 \end{array} \tag{7}$$

$$P = \begin{vmatrix} -\alpha_1 - \frac{1}{2}(T_{22} + T_{33}) - \frac{1}{3}T & \frac{1}{2}T_{12} & \frac{1}{2}T_{13} \\ \frac{1}{2}T_{12} & -\alpha_2 - \frac{1}{2}(T_{11} + T_{33}) - \frac{1}{3}T & \frac{1}{2}T_{23} \\ \frac{1}{2}T_{13} & \frac{1}{2}T_{23} & -\alpha_3 - \frac{1}{2}(T_{11} + T_{22} - \frac{1}{3}T) \end{vmatrix}$$

N^* is obtained from N by mirroring with respect to the main diagonal (transformed matrix).

The curvature is determined by four independent parameters α_i, β_i and the tensor$T_{\alpha\beta}$.

Type II

$$M \equiv \begin{vmatrix} \alpha_1 + \frac{1}{2}(T_{11} - T_{44}) + \frac{1}{3}T & \frac{1}{2}T_{12} & \frac{1}{2}T_{13} \\ \frac{1}{2}T_{12} & \alpha_2 + \frac{1}{2}(T_{22} - T_{44}) + \frac{1}{3}T + 1 & \frac{1}{2}T_{23} \\ \frac{1}{2}T_{13} & \frac{1}{2}T_{23} & \alpha_2 + \frac{1}{2}(T_{33} - T_{44}) + \frac{1}{3}T - 1 \end{vmatrix}$$

$$N = \begin{vmatrix} \beta_1 & \frac{1}{2}T_{34} & -\frac{1}{2}T_{24} \\ -\frac{1}{2}T_{34} & \beta_2 & \frac{1}{2}T_{14} + e \\ -\frac{1}{2}T_{24} & -\frac{1}{2}T_{14} & \beta_2 \end{vmatrix} \quad \begin{matrix} \sum \alpha_i = 0 & \alpha_1 + 2\alpha_2 = 0 \\ \sum \beta_i = 0 & \beta_1 + 2\beta_2 = 0 \\ & e = \pm 1 \end{matrix} \tag{8}$$

$$P = \begin{vmatrix} -\alpha_1 - \frac{1}{2}(T_{22} + T_{33}) - \frac{1}{3}T & \frac{1}{2}T_{12} & \frac{1}{2}T_{13} \\ \frac{1}{2}T_{12} & -\alpha_2 - \frac{1}{2}(T_{11} + T_{33}) - \frac{1}{3}T - 1 & \frac{1}{2}T_{23} \\ \frac{1}{2}T_{13} & \frac{1}{2}T_{23} & -\alpha_2 - \frac{1}{2}(T_{11} + T_{22}) - \frac{1}{3}T + 1 \end{vmatrix}$$

N^* is equal to the transformed N. The curvature is determined by two independent parameters α_i and β_i and the tensor $T_{\alpha\beta}$

Type III

$$M = \begin{vmatrix} \frac{1}{2}(T_{11} - T_{44}) + \frac{1}{3}T & \frac{1}{2}T_{12} + e_1 & \frac{1}{2}T_{13} \\ \frac{1}{2}T_{12} + e_1 & \frac{1}{2}(T_{22} - T_{44}) + \frac{1}{3}T & \frac{1}{2}T_{23} \\ \frac{1}{2}T_{13} & \frac{1}{2}T_{23} & \frac{1}{2}(T_{33} - T_{44}) + \frac{1}{3}T \end{vmatrix}$$

$$N = \begin{vmatrix} 0 & \frac{1}{2}T_{34} & -\frac{1}{2}T_{24} \\ -\frac{1}{2}T_{34} & 0 & -\frac{1}{2}T_{14} + e_2 \\ -\frac{1}{2}T_{24} & -\frac{1}{2}T_{14} + e & 0 \end{vmatrix} \tag{9}$$

$$P = \begin{vmatrix} -\frac{1}{2}(T_{23} + T_{33}) - \frac{1}{3}T & \frac{1}{2}T_{12} - e_1 & \frac{1}{2}T_{13} \\ \frac{1}{2}T_{12} - e_{13} & -\frac{1}{2}(T_{11} + T_{33}) - \frac{1}{3}T & \frac{1}{2}T_{23} \\ \frac{1}{2}T_{13} & \frac{1}{2}T_{23} & -\frac{1}{2}(T_{11} + T_{22}) - \frac{1}{3}T \end{vmatrix}$$

e_1 and e_2, which are equal to ± 1, are independent of each other. The curvature is determined by the tensor $T_{\alpha\beta}$.

It follows from (7), (8), (9) that the curvature of space-time in an orthreper is determined by the energy-momentum tensor and the invariants α_i, β_i (for type III only by the energy-momentum tensor). In addition, these formulas suggest their concretization by a more thorough study of the energy-momentum tensor.

§2. Classification of energy-momentum tensors

Let us consider parasymmetric tensors at a given point in space-time $T_{\alpha\beta}$ and $g_{\alpha\beta}$ ($|g_{\alpha\beta}| \neq 0$) and the corresponding λ-matrix

$$(T_{\alpha\beta} - \lambda g_{\alpha\beta}).$$

We will also keep in mind the fact that the space-time metric is defined by the form $g_{\alpha\beta}dx^\alpha\ dx^\beta$ of hyperbolic type (signature at a point of the form $---+$). Using the theorem obtained by the author [3; 1; §20], we can write out all possible characteristics of elementary divisors and then, using only real coordinate transformations, write the canonical forms of tensors $T_{\alpha\beta}$ and $g_{\alpha\beta}$ in some uniquely defined real quasi-orthreper (the word «quasi» is used here in the sense that among the coordinate vectors of such a reference point, isotropic vectors can also appear). It seems that in principle the following types of characterization of λ-matrices are also possible: $(T_{\alpha\beta} - \lambda g_{\alpha\beta})$:

$$1).[\mathrm{IIII}],\quad 2).[211]\quad 3).\ 31],\quad 4).\ [22],\quad 5).\ [4] \tag{6}$$

Moreover, these characteristics can also be subdivided into special sub-cases that can occur if some bases of elementary divisors are multiples, for example, [(21) 1] elementary divisors of the form $(\lambda - \lambda_1)^2$, $(\lambda - \lambda_1)$, $(\lambda - \lambda_2)$ and t. e. In addition, due to the uncertainty of the metric form, complex elementary divisors and *isotropic and complex* eigenvectors of the tensor $T_{\alpha\beta}$ may appear.

However, the fact that the metric signature looks like $(---+)$, reduces the number of possible types to three. This result is most easily obtained as a simple corollary of general formulas that give the canonical form a simple corollary of general formulas that give the canonical form of tensors in a quasi-orthreper for a pair of *real* tensors $g_{\alpha\beta}$, $T_{\alpha\beta}$ for *real* transformations and a non-degenerate tensor $g_{\alpha\beta}$ ($|g_{\alpha\beta}| \neq 0$), obtained by the author in 1949 [3; 1 §9]. These formulas are valid for any n-number of space dimensions and any metric signature and are included in the formulation of the following theorem. Matrices of two real symmetric tensors g_{ij} and T_{ij} under the condition $|g_{ij}| \neq 0$ can be reduced by a real nonsingular linear transformation of coordinates to the form:

$$(g_{ij}) = \begin{pmatrix} G_1 & & \\ & \ddots & \\ & & G_v \end{pmatrix}, \quad (T_{ij}) = \begin{pmatrix} T_1 & & \\ & \ddots & \\ & & T_v \end{pmatrix},$$

where V is the number of elementary divisors λ is the matrix $(T_{ij} - \lambda g_{ij})$, if we count a pair of complex-conjugate divisors as one. The symbols G_i and T_i denote square matrix blocks that have the number of rows and columns equal to the exponent and the corresponding elementary divisor real. They have the form:

$$G_i \begin{vmatrix} & & e^i \\ & \iddots & \\ e_i & & \end{vmatrix}, \quad T_i = \begin{vmatrix} & & \lambda_i e_i & e_i \\ & \iddots & \iddots & \\ \lambda_i e_i & e_i & & \end{vmatrix}, \quad e_i = \pm 1 \tag{7}$$

In the case of complex elementary divisors, for each complex divisor $(\lambda - \alpha - i\beta)^i$, there is a complex-conjugate $(\lambda - \alpha + i\beta)^i$ and, which, when combined together, define quadratic blocks of an even number of dimensions of the form:

$$G_{2i} = \begin{vmatrix} & & \boxed{\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}} \\ & \iddots & \\ \boxed{\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}} & & \end{vmatrix}, \quad T_{2i} = \begin{vmatrix} & & & \boxed{\begin{matrix} \alpha - \beta, & \alpha + \beta \\ \alpha + \beta, & -\alpha + \beta \end{matrix}} \\ & & \iddots & \boxed{\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}} \\ & \iddots & \iddots & \\ \boxed{\begin{matrix} \alpha - \beta, & \alpha + \beta \\ \alpha + \beta, & -\alpha + \beta \end{matrix}} & \boxed{\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}} & & \end{vmatrix} \tag{8}$$

The most important point in this argument is that the canonical form conversion is performed on *real path*, and not over the field of complex numbers, when the question of the signature becomes meaningless.

Thus, the characteristic λ of the matrix $(T_{ij} - \lambda g_{ij})$ determines the conditions of the signature, since the form G_i to some extent determines the signature of the metric. In addition, you should consider all possible cases of multiplicity of the roots of. This kind of research was conducted by the author, and then T. E. Izotov [4] and V. I. Golikov. Its essence is that the matrix $(g_{\alpha\beta})$ is reduced by real transformations to a diagonal form, after which the question of which e_i signature is possible is analyzed $(---+)$. If it is not possible for any choice of e_i, the case disappears. Omitting this obvious argument, we present the final result: *for gravitational fields, only three types of energy-momentum tensors are possible, determined by the characteristics, respectively:* I. [IIII]; II.]112], III. [13]. As for the characteristics of the form [22] and [4], they are generally impossible for a hyperbolic signature. These three types can be further classified in more detail, depending on whether there are multiple elementary divisors (m.e. with the same bases), and whether there are complex among them.

This detailed classification will look like this.

Type I

a) elementary divisors are real. Depending on the multiplicity of bases, sub-cases are possible α_1 ([IIII], α_2) [(II), II], α_3)[III) I], α_4)[(II) (II)], α_5)[(IIII)]. In an orthreper constructed on the basis of eigenvectors, it will have canonical forms

$$(g_{\alpha\beta}) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{pmatrix}, \quad T_{\alpha\beta} = \begin{pmatrix} -\lambda_1 & & & \\ & -\lambda_2 & & \\ & & -\lambda_3 & \\ & & & +\lambda_4 \end{pmatrix} \tag{9}$$

Here λ_i – *bases* of elementary divisors that match *with the eigenvalues* of the tensor T_{ij}. In the case of multiplicity of eigenvalues (the case of $a_2,\ a_3,\ a_4,\ a_5$), the corresponding λ_i should be equated to each other in (13).

b) there are complex elementary divisors. In this case, only two sub-cases are possible: b_1) [IIII] — there are two real and two complex-conjugate eigenvalues of the tensor T_{ij}, which in the real space has two eigendirections and one invariant 2-plane. b_2 [(II) II] has two complex-conjugates and one real (multiplicities of two) eigenvalues. The tensor T_{ij} has in real space a two-dimensional sheaf of eigendirections and one invariant 2-plane. Canonical formulas for $g_{\alpha\beta}$ and $T_{\alpha\beta}$ are derived from [11] and [12] in the real quasi-benchmark.

$$(g_{\alpha\beta}) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & 1 \\ & & 1 & \text{-1} \end{pmatrix}, \quad (T_{\alpha\beta}) = \begin{pmatrix} -\lambda_1 & & & \\ & -\lambda_2 & & \\ & & \boxed{\begin{matrix} \alpha-\beta, & \alpha+\beta \\ \alpha+\beta, & -\alpha+\beta \end{matrix}} \end{pmatrix} \tag{14}$$

moreover, in the case of (b_2), $\lambda_1 = \lambda_2$. It is not difficult to divide the real linear transformation that leads $(g_{\alpha\beta})$ in (14) to the diagonal form.

type II, [112]. As above, it is easy to check that only real eigenvalues are possible here. Depending on their multiplicity, we obtain the following subcases: 1). [112]; 2) [(II) 2]; 3) [1(12)]; 4) [(112)].
In the first case, we uniquely define an orthreper for which the canonical forms are obtained

$$(g_{\alpha\beta}) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad T_{\alpha\beta} = \begin{pmatrix} -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & -(\lambda_3+1) & 0 \\ 0 & 0 & 0 & \lambda_3-1 \end{pmatrix}. \tag{15}$$

Here we have three eigenvectors of the tensor $T_{\alpha\beta}$, one of which *isotropic* and one invariant 2-plane passing through the isotropic eigendirection. In the case of (2), the formulas (15) should contain $\lambda_1 = \lambda_2$. An integer 2-plane of eigenvectors appears, which is the linear shell of two non-isotropic eigenvectors corresponding to multiples of the elementary divisors. In the case of (3), put $\lambda_2 = \lambda_3$ in (15). There is a three-dimensional bundle of eigendirections spanned by three independent eigenvectors.

Type III [13]. *Only proper real numbers are possible.* Depending on the multiplicity of eigenvalues, two possibilities arise: 1) [13], 2) [(13)].
In the first case, there are two eigenvectors, one of which is non-isotropic (corresponding to a simple elementary divisor). The other, isotropic, corresponds to a non-simple elementary divisor. There is a 3-dimensional invariant sheaf passing through an isotropic eigendirection. It is always possible, and unambiguously, to define an orthreper for which

$$(g_{\alpha\beta}) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad T_{\alpha\beta} = \begin{pmatrix} -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2-1 & 0 & 0 \\ 0 & -1 & -\lambda_2 & -1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}. \tag{16}$$

In the case of (2) and (16), put $\lambda_1 = \lambda_2$. Here we have an invariant 2-plane spanned by isotropic and non-isotropic eigendirections, all consisting of eigenvectors.

This completes the complete classification of possible energy-momentum tensors that characterize the state. [5, §31, 32, 33].

1. `Energy-momentum tensor of pulverized matter` consisting of non-interacting uncharged particles. $T_{\alpha\beta} = \varepsilon_0 U_\alpha U_\beta$, where ε_0 is the rest energy of the particle,$\varepsilon_0 = \mu_0 C^2$, μ_0 rest mass, U^α – unit vector 4-particle velocity. In this case, we have a characteristic of the simple type [I(III)] with real eigenvalues, which corresponds to one nonzero number equal to the rest energy in the given companion system, and three — equal to zero.

1. `The case of an ideal liquid` is defined as when there are uncharged particles exerting pressure on each other according to Pascal's law for liquids.

$$T_{\alpha\beta} = (\varepsilon - p)U_\alpha U_\beta + p g_{\alpha\beta}.$$

The characteristic has the form [I (III)]. The case differs from the previous one only in that there is one eigenvalue equal to the difference between the field energy and pressure p. In addition, there are three more eigenvalues equal to the pressure.

3. `The energy-momentum tensor of an electromagnetic field in a void` in the absence of charged particles.

$$T_{\alpha\beta} = -\frac{1}{4\pi}(F_{a\delta}F^\delta_\beta - \frac{1}{4}F_{\delta\tau}F^{\delta\tau}g_{\alpha\beta}),$$

where$T^\delta\delta = 0$, $F_{\alpha\beta}$ – is the electromagnetic field tensor. This is where the characteristics of both simple and non-simple types appear:
1) [(II) (II)] – a field without a singularity
2) [II2] – a singular case corresponding to pure radiation.
In the first case, you can define an orthreper with respect to which

$$(g_{\alpha\beta}) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad (T_{\alpha\beta}) = \begin{pmatrix} -\lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & -\lambda \end{pmatrix},$$

and in the second, respectively:

$$(g_{\alpha\beta}) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad T_{\alpha\beta} = \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \boxed{\begin{matrix} e & -e \\ -e & e \end{matrix}} & \end{pmatrix} \quad \begin{matrix} T_{\alpha\beta} = e l_\alpha\, l_\beta,\ l_\alpha\, l^\beta = 0 \\ S_p\, T_{\alpha\beta} = 0,\ e = \pm 1 \end{matrix} \tag{17}$$

It is not difficult to consider more complex cases of the energy-momentum tensor that correspond to a more complex structure of the state of matter. For example, given dissipative processes, thermal conductivity, or elastic forces, you can get a more complex characteristic and write down the canonical form of the energy-momentum tensor based on (11) and (12).

The following fact is fundamental: *in the framework of general relativity, any possible physical state of a system is described by an energy-momentum tensor of the form* (13) – (16).

The case of complex eigenvalues has not actually been studied so far. It is very important to note that the results obtained above, although written locally in an orthreper, are essentially invariant and can be written in tensor form for the *region of space-time.*

1. §3. Problem statement. The case of an electromagnetic field.

When solving any problem in the framework of general relativity, it is necessary, regardless of the method used, to carry out two stages of research:

1) *Find properties common to any energy-momentum tensor and any type of gravity field* (such as the conservation law of covariant divergence of the energy-momentum tensor or the Bianchi identity), and a more complex stage –
2) *define properties related to the specific issue.* They will always lead to the need to find out the type of field and the type of energy-momentum tensor. If different types of these two classifications are possible for a given problem statement, then naturally the question arises of dividing the problem into mn cases, where m – is the number of field types allowed by the problem statement, and n – is the number of allowed types of the energy-momentum tensor. In *general relativity* (aside from sub-cases corresponding to multiplicities of eigenvalues, there are *no more than nine types of possible physical states* (3 types of fields $\times$ 3 types of $T_{\alpha\beta}$).

The problem of determining these types is fraught with difficulties and requires a fine mathematical apparatus and considerable ingenuity to solve it. However, this problem is fundamental for constructing a deeper theory and is solved implicitly by many physicists for particular cases.

When solving this theorem, first of all, the question arises : is there not some correspondence between the three types of fields (§1) and the three types of energy-momentum tensors (§2) that excludes some logical intersections of these two classifications? There is no general answer to this question yet, but the study of particular cases suggests that, generally speaking, different types of classification can correspond to one type of one classification.

For this purpose, let us consider, for example, the «singular» case of pure electromagnetic radiation in a void defined by the formulas (17). It is not difficult to verify that in this case

$$T_{\alpha\beta} = el_\alpha\, l_\beta, \quad l_\alpha\, l^\beta = 0, \quad e = \pm 1,$$

where l_β is an isotropic eigenvector $T_{\alpha\beta}$. The electromagnetic field tensor $F_{\alpha\beta}$ can be represented as

$$F_{\alpha\beta} = 2(\rho l_{[\alpha} h_{\beta]} + \sigma l_{[\alpha} k_{\beta]}), \tag{18}$$

where h_α and k_β are two vector non-isotropic fields, and the scalars ρ and σ represent the sine and cosine of some scalar, since

$$\rho^2 + \sigma^2 = 4\pi,$$

moreover, in the orthreper

$$(F_{\alpha\beta}) = \begin{pmatrix} 0 & -\rho & 0 & 0 \\ \rho & 0 & \sigma & 0 \\ 0 & -\sigma & 0 & \alpha \\ -\rho & 0 & -\sigma & 0 \end{pmatrix}.$$

For this coordinate system, let the vector fields defining the electromagnetic field tensor $F_{\alpha\beta}$ have the form

$$L_\alpha = A\delta^4_\alpha, \quad h_\alpha = B\delta^2_\alpha, \quad k_\alpha = C\delta^3_\alpha. \tag{20}$$

We require that the Einstein field equations (1) are satisfied, which in this case ($T = 0$) will have the form

$$R_{\alpha\beta} = el_\alpha\, l_\beta, \tag{21}$$

and Maxwell's equations in the void

$$F_{[\alpha\beta,\ \gamma]} = F^\sigma_{\alpha\ ,\sigma} = 0. \tag{22}$$

Then we can verify by direct verification that, assuming

$$\alpha = M(px^2 + nx^3 + S) \qquad \partial_1\rho = \partial_1\sigma = 0$$
$$\beta = M(x^3 + \int p dx^1 + v) \qquad B\partial_3\rho - C\partial_2\sigma = 0$$
$$\gamma = M(x^2 + \int n dx^1 + \omega) \qquad C\partial_2\rho + B\partial_3\sigma = 0,$$

where $A(x^4)$, $M(x^4)$, $n(x^1)$, $v(x^4)$, $S(x^1)$, $\omega(x^4)$, $p(x^1)$ — arbitrary functions, and $B(x^4)$, $C(x^4)$ is defined from the equations:

$$BB' = \frac{B''}{B'} - \frac{M'}{M} = CC'(\frac{C''}{C'} - \frac{M'}{M}) = -\frac{L}{2}A^2B^2.$$

This satisfies the field equations (21), and Maxwell's equations (22), and by examining the Weyl tensor of conformal curvature, it is easy to see that `field will be of type 1`.

Further, if we assume that

$$\begin{aligned} &\alpha = M(v^1x_2 + t^1x^3 + S) && \partial_1\rho = \partial_1\sigma = 0 \\ &\beta = M(v + C_1x^3l^{x^4}p) && B\partial_3\rho - C\partial_2\sigma = 0 \\ &\gamma = M(t + C_1x^2l^{x^4} + q) && C\partial_2\rho + B\partial_3\sigma = 0, \end{aligned}$$

where $A(x^4)$, $M(x^4)$, $v(x^1)$, $t(x^1)$, $S(x^1)$, $p(x^4)$, $q(x_4)$ — arbitrary functions, and $B(x^4)$ and $C(x^4)$ is defined from the equations

$$B'' - \frac{M'}{M}B' + \frac{L}{2}A^2B = C'' - \frac{M''}{M}C' + \frac{L}{2}A^2C = 0,$$

then the equations (21) and (22) will again be satisfied, and the study of the Weyl tensor leads to the conclusion that the field will be of type II.

In the case of a type III field, we restrict ourselves to stating a theorem that is of basic importance for studying the electromagnetic field in this formulation of the question: *the electromagnetic radiation vector $l_\alpha(x)$ for a type 3 field is the eigenvector of the energy-momentum tensor and has the properties:* 1) defines *geodesic*($l_{\alpha,\beta}\ l^\beta = f\ l_\alpha$); 2) *isotropic*($l_\alpha\ l^\alpha = 0$ и 3) *conformal-gradient* ($\varphi\ l_\alpha$ – *deg)* congruence. In addition, in the case of 3 types, it is always possible to normalize the field l_α so that the Weyl tensor is expressed in terms of the vector fields l_α, h_α, k_α, which define the electromagnetic field, in in the form of:

$$C_{\alpha\beta\gamma\delta} = -4(l_{[\alpha}h_{\beta]}l_{[\gamma}v_{\beta]}l_{[\alpha}v_{\beta]}l_{[\gamma}h_{\delta]} + l_{[\alpha}k_{\beta]}k_{[\gamma}h_{\delta]} + k_{[\alpha h\beta]}l_{[\gamma}k_{\delta]}),$$

where v_α is a field defined by the following conditions

$$v_\alpha\ l^\alpha - 1 = v_\alpha\ h^\alpha = v_\alpha\ k^\alpha = v_\alpha v^\alpha = 0.$$

Thus, even in the fairly simple case of electromagnetic radiation, all three types of fields are possible. Without dwelling on the physical interpretation of the results obtained, which requires a special article, we note only that the solution of the problem formulated above in this case allows us to obtain quite interesting results related to the integration of a complex system of Einstein field equations.

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