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О ТЕОРЕМЕ БИРКГОФА (НА АНГЛИЙСКОМ) *

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ON BIRKHOFF'S THEOREM (IN ENGLISH)

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One of the most important special cases of gravity fields is the field with central symmetry. At the same time, when solving many specific tasks, we have to consider *static fields.*

As for the question of centrally symmetric fields in empty space, which are not necessarily static, their existence was rejected on the basis of the following statement, first expressed by Birkhoff [1]: «*Every centrally symmetric field in the void — static*» and is therefore defined by the Schwarzschild metric. This statement has since become generally accepted and has been included in all serious monographs on general relativity [2-8], with the exception, apparently, only of the famous book by Fock [13]; it was followed by the earlier author of the article [9]; it is often used in relativistic theory. cosmology and mechanics to justify responsible conclusions. However, this statement is erroneous.

In this paper, we give *a strict solution of the equations of a centrally symmetric field in the void*

$$R_{\alpha\beta} = 0, \tag{1}$$

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and along the way, the question becomes clear when such a solution will be static, which refutes the above statement by Birkhoff.

Note that the criterion of correctness of this statement, as it was discovered after writing the paper, is contained in [14], where it is noted that it is impossible to prove Birkhoff's theorem without violating the well-known conditions of Lichnerovich continuity [15]. This article provides a general solution to the problem and confirms these preliminary considerations directly. Some authors also use limit conditions to justify Birkhoff's position, but this does not change the position, as will be shown in §4.

The solution of the above problem is based on the theory of *confor-reducible* gravity fields and some of its results obtained earlier by the author.

§1. Centrally symmetric fields

As is known, the metric of a centrally symmetric (non-static) field can be written as [5, стр. 317] [9, стр. 210]

$$ds^2 = Adx^{1^2} = 2Bdx^1dx^4 + C(dx^{2^2} + \mathrm{Sin}^2x^2dx^{3^2}) + Ddx^{4^2}, \tag{2}$$

where A, B, C, D – some functions are from $x^1,\ x^4$, and the metric is defined exactly up to coordinate transformations.

$$x^{1'} = f(x^{1'}, x^4), x^{2'} = x^2 + k\pi, x^{3'} = x^3 + s\pi, x^{4'} = \theta(x^1, x^4), \tag{3}$$

where $k,\ s$ are integers, and $f,\ \theta$ — *arbitrary* functions of their arguments.

A strict mathematical definition of any centrally symmetric field is that a linear element defined by such a field admits a 3-term group of motions, which has the same structure as the group of rotations in three-dimensional Euclidean space. This is [9] – the so-called *undecidable* group G_3 with an operator structure.

$$[X_1X_2] = X_3,\quad [X_2X_3] = X_1,\quad [X_3X_1] = X_2,$$

where $X_{sf} \overset{\text{def}}{=} \xi_s^\alpha\partial_\alpha (s = 1, 2, 3)$ and for the coordinate system in which the metric (2) takes place, as you can easily see, the vectors ξ_s^α have the form:

$$\begin{aligned}
&\xi_1^\alpha = \mathrm{Cos}\ x^3\delta_2^\alpha - \mathrm{Sin}\ x^3\ \mathrm{ctg}\ x^2\delta_3^\alpha,\\
&\xi_2^\alpha = \mathrm{Sin}\ x^3\delta_2^\alpha + \mathrm{Cos}\ x^3\ \mathrm{ctg}\ x^2\delta_3^\alpha,\\
&\xi_3^\alpha = \delta_3^\alpha.
\end{aligned} \tag{4}$$

Note that the metric (2) can be obtained strictly unambiguously, based on the given*invariant-group* definition of the concept of a centrally-symmetric field, and show that the coordinate systems for which (2) and (4) is defined quite correctly and [9, page. 209-210]. On the contrary, if we proceed from the metrics (12), considering it as the definition of such a field, then the structure, the form of the Killing vector (4), is uniquely determined. It is proved that in general the metric does not admit a group of movements other than the group (4). This confirms that the definition of central symmetry given above is invariant-group and therefore does not depend on the choice of coordinate system; it is equivalent to other definitions given in books on general relativity.

Using arbitrary functions f and θ in valid transformations (3), you can divide the components of $g_{\alpha\beta}(x)$ metrics (2) impose two more conditions that simplify the appearance of the metric. These conditions, however, are not entirely arbitrary, since they will only be mathematically correct if they satisfy the following three conditions: 1). Transformations (3) must be *non-degenerate* (this is written by the condition: Jacobian of the transformation $\partial_1 f\ \partial_4\theta - \partial_4 f\ \partial_1\theta \neq 0$); 2). It must also be *real* (since otherwise the meaning of the concept of *metric signature* and, in particular, the concept of static); 3) would be lost. The conditions imposed during transformations (3) must not contain any *additional* assumptions about the components of the metric $g_{\alpha\beta}(x)$.

The latter condition is particularly important, since errors in the «proofs» of Birkhoff's statement are mostly related to the violation of this condition. For example, it would not be possible to require that the components of g'_{22} (in the new coordinate system), as is often done, met the requirements of $g'_{22} = -(x^{2'})^2$, since it is easy to calculate that $g'_{22} \equiv g_{22}(x^\alpha(x^{\beta'})) \equiv C(x^\alpha)$, and you can specify many cases where the requirement specified here cannot be satisfied if it was not assumed that C has well-defined properties that do not follow from the definition of a centrally symmetric field (for example, if $C = \text{const}$, then the requirement is $g'_{22} = -(x^{1'})^2$ impossible). Moreover, you can strictly show [2–5], what is the requirement of $g'_{22} = -(x^{1'})^2$ is strictly equivalent to the condition that empty space-time is static.

Among the various possible correct requirements, you can specify the following two that meet all three of the above conditions: [9, pp. 211]:

$$g'_{14} = 0, \quad g'_{11}g'_{44} = -1.$$

By performing just such a coordinate transformation, we will, of course, again get a metric of the form (2), where, however, we can put $B = 0,\ AD = -1$.

After that, putting C out of the bracket, we write this metric in the form

$$ds^2 = C(x^1,\ x^4)[\mathrm{I}(x^1,\ x^4) + \mathrm{II}(x^2,\ x^3)], \tag{5}$$

where are the quadratic binary forms

$$\mathrm{I}(x^1,\ x^4) \stackrel{\text{def}}{=} \frac{A}{C}dx^{1\,2} - \frac{1}{AC}dx^{4\,2}, \quad \mathrm{II}(x^2,\ x^3) \stackrel{\text{def}}{=} dx^{2\,2} + \sin^2 x^2 dx^{3\,2}, \tag{6}$$

each depends on two variables that are different for each form and have, respectively, signatures of the form $(+-),\ (--)$.

Such fields can be called *conformally reducible* and in a special coordinate system, the general form of the metric of such a field due to *binary* transformations can be written as

$$ds^2 = v^2[e(\gamma^2 dx^{1\,2} - dx^{4\,2}) - (dx^{2\,2} + \beta dx^{3\,2})], \tag{7}$$

where v is an arbitrary function $x^1,\ x^2,\ x^3,\ x^4$; γ depends only on$x^1,\ x^4$; β depends on $x^2,\ x^3$; $e = \ pm1$ (in this connection, either $x^2,\ x^3$; $e = \pm 1$ can play the role of coordinate time, and it is convenient not to fix the value of e in advance).

It follows from (5) that *every centrally-symmetric field is a conformally-reducible field of the special form.* A general solution to the problem of solutions of equations of a conformally-reducible field has not yet been obtained, but in the case when solutions of the field in the void are considered ($R_{\alpha\beta} = 0$) or $R_{\alpha\beta} = \varkappa g_{\alpha\beta}$, there is a general solution to the question [9, §46]. The author of the article gives a statement of the question, a method for solving it, and a study of some of the possible cases. A generalization for the field equations $R_{\alpha\beta} = \varkappa g_{\alpha\beta}$ ($\varkappa \neq 0$) and an analysis of the cases not considered are given in [11]. Thus, for centrally-symmetric fields in *void*, it becomes possible to apply general results that hold for conformally-reducible fields in the void.

§2. Centrally symmetric fields in the void

For any conformally reducible space-time in the void, the following properties can be formulated with the metric (7) in mind:

1) the v function in (7) looks like

$$v^{-1} = \psi(x^1,\ x^4) + \varphi(x^2,\ x^3);$$

Depending on the type of functions $\varphi,\ \psi,\ \beta,\ \gamma$, there are eight different possible cases. In order to determine which of these cases is possible for a centrally symmetric field in the void, we note the

specifics of the form (5), under the conditions (6). Compared to (7) and (8), this metric has the following features:

$$(1)\ \psi^2 = \frac{1}{c^2},\ \varphi = 0 \qquad (2)\ \beta = \sin x^2.$$

These two conditions determine the possible [one of the eight] solution type unambiguously [9, page 397]. Thus, we have the following result (instead of Birkhoff's statement): *the general form of the metric of a centrally symmetric field in empty space-time, in a special coordinate system, can be written as:*

$$ds^2 = \frac{1}{\psi^2}[e(\gamma^2 {dx^1}^2 - {dx^4}^2) - {dx^2}^2 \sin^2 x^2 {dx^3}^2], \tag{8}$$

where the functions $\psi,\ \gamma$ depend only on the variables $x^1,\ x^4$ and are determined by integro-differential conditions.

$$\begin{aligned} &\psi_1 = \nu\gamma \\ &\psi_4^2 = c_1\psi^3 + e\psi^2 + \nu^2 \qquad (\psi_i \equiv \frac{\partial\psi}{\partial x_i}) \\ &\gamma = \psi_4\left(\nu^1 \int \frac{dx^4}{\psi_4^2} + \lambda\right) \end{aligned} \tag{9}$$

Here $\nu(x^1),\ \lambda(x^1)$ – *arbitrary* functions from $x^1,\ e = \pm 1$.

The equations (9) are not generally integrated in elementary functions, and only with some additional requirements imposed on $\psi,\ \gamma$, various specific solutions of the field equations can be obtained from here.

For example, the well-known Schwarzschild solution can be obtained from (9) for $e = 1$ (and only in this case), when the role of `coordinate time` is played by the variable x^1, and x^4 is a generalized radius-vector. By putting in (8) and (9)

$$\begin{aligned} &e = 1,\ \ x^1 \to x^4,\ \ x^4 \to x^1, c_1 = -2m \\ &\psi = \frac{1}{f},\ \ \gamma = \pm\left(\frac{f-2m}{f^3}\right)^{1/2},\ \ f = \frac{c_1}{2}e^{x^1} + \frac{m^2}{c_1}e^{-x^1} + m, \end{aligned}$$

it is easy to verify by selecting $\lambda,\ \nu$ that the conditions (9) are fulfilled identically to. After that, performing coordinate transformations (always obviously possible)

$$x^{1'} = f,\ \ x^{2'} = x^2,\ \ x^{3'} = x^3,\ \ x^{4'} = x^4,$$

let us reduce the metric (8) to the classical form of the Schwarzschild metric in spherical coordinates:

$$ds^2 = -A^{-1}{dx^1}^2 - {x^1}^2({dx^2}^2 + \sin^2 x^2 {dx^3}^2) + A\ {dx^4}^2 \tag{10}$$

$$A = 1 - \frac{2m}{x^1}.$$

If we take in (8) $e = -1$, then the variable x^4 will play the role of time, and it is easy to see that in this case, for any values of $\gamma,\ \psi$, the Schwarzschild metric cannot be obtained. We can make the following rough estimate of the solution formulated above in comparison with the Schwarzschild metric (estimate of the error made by Birkhoff). If we denote the entire class of valid solutions as one, then Birkhoff first lost exactly half of ($e = -1$) and then allowed ∞ solutions that depend on the function of a single variable.

§3. Static condition

As you know, a field is called *static* if there is a coordinate system in which $g_{i4} = 0$ ($i = 1,\ 2,\ 3$) and $g_{\alpha\beta}$ do not depend on the variable x^4, which plays the role of coordinate time [metric signature $---+$]. This essentially non-invariant definition can be expressed in an invariant way using the apparatus of group theory. Let us consider, in particular, a centrally symmetric static solution in the void. Its metric must be the Schwarzschild metric, and, as is immediately clear, in this case, in addition to the motion

operators (4) with the structure G_3 rotations, it also allows the operator $X_4 = \xi_4^\sigma \partial_\sigma$, where $\xi_4^\sigma(x) = \delta_4^\alpha$, and hence the static condition can be invariantly defined by the following conditions: 1) the field admits a group of moves G_4 with the structure [9. §§10, 49]

$$[X_1X_2] = X_3, \quad [X_2X_3] = X_1, \quad [X_3X_1] = X_2$$
$$[X_iX4] = 0 \qquad (i = 1,\ 2,\ 3). \tag{11}$$

Moreover, the norm of the vector ξ_4^α, which defines the fourth operator (static operator)

$$g_{\alpha\beta}\xi_4^\alpha \ \ \xi_4^\alpha > 0 \tag{12}$$

(for a signature of the form $---+$).
2). The metric also allows a discrete group `mirror images` $x^4 \to -x^4$. The conditions (11) and (12) obviously do not depend on the choice of the coordinate system and are therefore invariant as a necessary (but not sufficient, in general) condition that a centrally symmetric field in the void is also static; it is necessary, in any case, to require that it also allows the fourth static operator.

In order to show that Birkhoff's statement is incorrect, we proceed as follows.

Consider the metric (8) for $e = -1$ and show that it admits the group G_4 with the structure (11), but, in general, does not satisfy the necessary condition (12).

We define the form of the fourth operator from the equations of the structure (11), calculating the commutators $[X_i, X_4] = 0$ for $i = 1,\ 2,\ 3$ and keeping in mind that these the three operators are defined by the formulas (4). We get directly from this that

$$\xi_4^\alpha(x) = \xi_4^1(x^1,\ x^4)\delta_1^\alpha + \xi_4^4(x^1,\ x^4)\delta_4^\delta, \qquad \xi_4^2 = \xi_4^3 = 0.$$

In order for this operator to determine motion, it is necessary and sufficient that it satisfies the Killing equations [11, §51]

$$\xi_4^\sigma \partial_\sigma g_{\alpha\beta} + g_{\alpha\beta}\ \partial_\beta \xi_4^\sigma + g_{\beta\sigma}\ \partial_\alpha \xi_4^\sigma = 0. \tag{13}$$

Writing these equations for the metric (8) under the conditions (9), we find that the vector ξ_4^α will have the form:

$$\xi_4^\alpha = -\frac{\nu\psi_4}{\psi_1}\delta_1^\alpha + \nu\delta_4^\alpha, \qquad \nu_1 \neq 0, \tag{14}$$

where $\nu,\ \psi$ are defined by the conditions (9) for $e = -1$. In this case, the equations [13] for this vector are satisfied due to the conditions (9) and their consequences.

This raises the question: in addition to the (14) operator, does the space also allow other operators, including the static operator? The author shows that, firstly, every conformally-reducible field in the void is a field of type I according to the classification of fields proposed by the author [9, §19], and, secondly, that if a field of type I [in the void] admits a group of motions $G_r,\ (r > 4)$, then it will be flat with the Minkowski metric [9, §28, 46]. Since the field we are considering is not flat, the field does not allow any other operators other than (14). In addition, in principle, the group operator is any linear combination of operators of some basis of the group. It is not difficult to see that for the structure (11), when the Killing vector can only have coordinates $\xi_4^1,\ \xi_4^4$, any linear combination with vectors (4) forbidden. Therefore, the only possible fourth operator here is $\xi^\alpha(x)$ will only have the coordinates (14). But it is not difficult to verify that, generally speaking, the necessary condition (12) is not met. Namely, by writing (12) for $e = -1$, we get

$$-\frac{\gamma^2}{\psi^2}\left(\frac{\nu\psi_4}{\psi_1}\right)^2 + \frac{\nu^2}{\psi^2} > 0.$$

Replacing $\psi_1,\ \psi_2$ in this inequality by the formulas (9) and dividing it by exact squares, we obtain the condition:

$$C_1\ \psi < 1.$$

In any four-dimensional region of space-time, where ψ is continuous, it is bounded. Therefore, by choosing the constant C_1 (which is arbitrary), it is possible to ensure that the inequality did not

occur in this region. Therefore, ξ_4^α cannot define the static operator in this case. Birkhoff's statement does not hold. The same fact can be established less sparingly by showing that there are no coordinate transformations that translate the (8) metric into the Schwarzschild metric, provided that the vectors (4) retain their form, and the vector ξ_4^α goes to $\xi_4^{\alpha'} = \delta_4^{\alpha'}$. We omit this trivial argument.

§4. Some corollary and discussion of the result

The existence of non-static centrally symmetric fields in the void means that, for example, in the case of an island centrally symmetric distribution of matter, non-static fields can be created in the *outer* region of space-time. *Centrally symmetric pulsations of gravitating masses can create gravitational waves* – the conclusion is interesting not only for cosmological problems.

As noted above, sometimes the formulation of Birkhoff's statement requires additionally that the metric at «spatial» infinity ($r \to \infty$) tends to the Minkowski metric.

With regard to this requirement, it should first be noted that it makes sense only in a certain special coordinate system and therefore requires an invariant formulation of for serious consideration. In its invariant form, it means the following: as we approach a certain hypersurface (analogous to the hypersurface $r = \text{const} \to \infty$ in spherical coordinates, for the Schwarzschild metric), which is determined by physical requirements so that the field becomes weak as much as possible near it, the metric of a space differs as little as possible from the metric of a flat space (the curvature tensor must tend to zero). For a field with the metric (8), this statement is executed automatically. This can be confirmed first of all by the following argument. A gradual weakening of the field as we approach the hypersurface S means a decrease in the curvature of space-time, which always means an increase in the possibility of mapping space at the expense of itself (automorphism).

More precisely, weakening a field is equivalent to saying that its curvature tensor tends to zero, that is, its metric approaches that of a flat space. Therefore, approaching S, the space will differ arbitrarily little from the Minkowski space. Any field with the metric (8) is a type I field, according to the author's classification. Any field of type I, if its mobility exceeds G_4 – flat. In other words, the limit conditions take place automatically.

Otherwise, we will come to the same conclusion by calculating the components of the curvature tensor for the metric (8), under the conditions (9). It is easy to verify that all nonzero components of the curvature tensor, up to bounded multipliers, have the form

$$\frac{1}{\psi} \tag{15}$$

Hence, the role of the generalized radius-vector in this case is played here by the function ψ. The hypersurface S can be written as $\psi = C$ when $c \to \infty$. *With an unlimited increase in ψ , we will get a metric that is arbitrarily close to the metric of a flat space.*

The next question, which arises naturally in the study of the obtained solution, can be formulated as follows.

At large distances (in the wave zone), we will have a weak gravitational wave, which in small areas can be represented as a plane wave, which should be transverse. This conclusion, however, would seem to contradict the symmetry of the problem.

Here we can argue as follows. If we consider the field at small distances, then the field will not be weak at all. Linearization of the field equations is not a correct operation, and at the same time the conclusion about the transverseness of waves obtained by Einstein back in his time does not pass. If we study the field at large distances, then the conclusion about the frequency of waves, it would seem, takes place. However, at large distances, as shown above, the curvature of space becomes as small as possible, and the field will be as close to flat as possible - the waves are damped, they «almost» do not exist. This removes the contradiction. It can be argued that they only «almost» do not exist, but in fact they still exist. The answer is obvious. The space in the wave zone is static (non-wave) with the same degree of accuracy as the statement about the «weakness» of the field, which is associated with the rejection of

quadratic terms in the expansion, is necessary for a linearized theory and for the conclusion about the transverse nature of waves.

In any case, the field with the metric (8) at ($e = pm1$) undoubtedly deserves careful study.

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