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ИНВАРИАНТНОСТЬ ОТНОСИТЕЛЬНО РЕПАРАМЕТРИЗАЦИИ ВРЕМЕНИ И ЕЁ СВЯЗЬ С ФЕНОМЕНАМИ ТЁМНОЙ ЭНЕРГИИ И ТЁМНОЙ МАТЕРИИ

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Загадочная природа тёмной энергии (DE) бросает вызов нашему пониманию космоса, выступая в качестве силы, ответственной за ускоренное расширение Вселенной. В данной работе космологическая постоянная Эйнштейна (Λ_E) рассматривается как проявление DE, интерпретируемое в рамках репараметризационно-инвариантной масштабной симметрии (RISS). В этой парадигме Λ_E возникает как “кинетический"член, происходящий из относительного временного движения, что отличает его от традиционной кинетической энергии, основанной на относительном пространственном движении.

В центре данного исследования находится космический масштабный фактор $\lambda(t)$, ре-параметризационная конструкция, способная динамически наделять Λ_E значимостью в расширенных уравнениях общей теории относительности Эйнштейна (EGR). Путём тщательных выводов формулируются управляющие уравнения для $\lambda(t)$ и его взаимодействия с Λ_E. Наложение симметрии репараметризации на уравнения движения открывает глубокий путь к решению проблемы недостающей массы, проявляющейся на галактических и внегалактических масштабах. Здесь некорректные (не сопутствующие) временные параметризации порождают фиктивные силы, чьё присутствие согласуется в рамках симметрийного подхода, обеспечивая когерентное описание явлений.

Этот основанный на симметрии подход естественным образом приводит к MOND-подобной зависимости, Модифицированная Ньютоновская Динамика (MOND), $g \sim \sqrt{a_0 g_N}$, где g обозначает гравитационное ускорение, a_0 представляет собой фундаментальное ускорение MOND, а g_N — ньютоновское ускорение. Теоретические предсказания для Λ_E и a_0 демонстрируют поразительное соответствие их наблюдаемым значениям, что придаёт достоверность данной интерпретации. Это соединение концептуальной ясности и математической строгости подчёркивает потенциально объединяющий принцип в нашем понимании гравитационных явлений, связанных с тёмной энергией и тёмной материей, а также широкой космической ткани.

Ключевые слова: космология: теория, тёмная материя и энергия, космологическая постоянная Эйнштейна; гравитация: теория, модифицированная ньютоновская динамика (MOND), масштабно-инвариантный вакуум (SIV), репараметризационная инвариантность.

TIME REPARAMETRIZATION INVARIANCE AND ITS RELATION TO THE DARK ENERGY AND DARK MATTER PHENOMENA

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The enigmatic nature of dark energy (DE) challenges our comprehension of the cosmos, appearing as the force responsible for the Universe’s accelerating expansion. This work considers the Einstein Cosmological Constant

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(Λ_E) as a manifestation of DE, interpreted through the framework of Reparametrization Invariant Scaling Symmetry (RISS). Within this paradigm, Λ_E emerges as a "kinetic energy"term, derived from relative temporal motion, setting it apart from the conventional kinetic energy based on spatial relative motion.

Central to this exploration is the cosmic scale factor $\lambda(t)$, a reparametrization construct capable of rendering Λ_E dynamically as relevant to the extended equations of Einstein's General Relativity (EGR). Through meticulous derivations, the governing equations of $\lambda(t)$ and its interplay with Λ_E are articulated. Imposing reparametrization symmetry on the equations of motion reveals a profound avenue for addressing the missing mass problem evident at galactic and extragalactic scales. Here, improper (non co-moving) temporal parametrizations introduce fictitious forces, whose presence is reconciled through the symmetry framework, ensuring a coherent treatment of the phenomena.

This symmetry-based approach naturally yields the MOND-like relationship, Modified Newtonian Dynamics (MOND), $g \sim \sqrt{a_0 g_N}$, where g denotes gravitational acceleration, a_0 represents the fundamental MOND acceleration, and g_N is the Newtonian acceleration. The theoretical predictions for Λ_E and a_0 demonstrate remarkable alignment with their observed magnitudes, lending credence to this interpretation. This synthesis of conceptual clarity and mathematical rigor underscores a potential unifying principle in our understanding of dark energy and dark matter gravitational phenomena and the broader cosmic tapestry.

Keywords: cosmology: theory, dark matter and energy, Einstein cosmological constant; gravity: theory, Modified Newtonian Dynamics (MOND), Scale-Invariant Vacuum (SIV), Reparametrization Invariance.

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Introduction

Modern physics stands at the crossroads of profound inquiry, facing a paradox that has defied resolution for decades. Quantum Field Theory (QFT) and General Relativity (GR), pillars of our understanding of nature, have successfully described phenomena within the terrestrial and Solar System scales. Yet, when applied to the vast scales of galaxies and the cosmos, an unsettling disparity emerges. Observable matter—the tangible substance of stars, planets, and galaxies—accounts for a mere $\approx 5\%$ of the Universe's energy content. By contrast, Dark Energy (DE) dominates with $\approx 70\%$, and Dark Matter (DM) contributes $\approx 25\%$ according to the conventional ΛCDM model of cosmology [1].

Numerous hypotheses have been proposed to explain the nature of DE and DM. These range from exotic fields and particles to profound modifications of Einstein's theory of gravity [2–4]. Yet, despite intense efforts, direct detection of DE or DM has eluded physicists for over four decades, leaving the dominant explanations incomplete.

This work explores a novel resolution to the DE and DM enigmas through an extension of Einstein's General Relativity (GR). Specifically, it revisits and refines the geometric framework proposed by Weyl in 1918, which was famously criticized by Einstein himself [5]. Subsequent reformulations, notably those employing the Weyl Integrable Geometry (WIG), addressed many of these concerns [6,7], with significant contributions from thinkers like [8] and [9].

In the generalized Weyl geometry, additional structural elements complement the metric tensor $g_{\mu\nu}$. These include a vector of metrical connection, κ_μ, which relates to the change in length magnitude as $dl = l\kappa_\mu, dx^\mu$, and a gauge scale factor λ. Here, I adopt the French term "connexion"to avoid conflating the concept of metrical connection with the standard "connection"employed in Einsteinian General Relativity (EGR). Within the Weyl Integrable Geometry (WIG) framework, the connexion vector κ_μ becomes directly linked to the scale factor λ via $\kappa_\mu = -\partial_\mu \ln \lambda$.

The hypothesis advanced here is that the apparent anomalies observed at galactic and cosmological scales stem from deviations from Einstein's GR. Specifically, the GR framework assumes that lengths remain invariant ($dl = 0$) during parallel transport. This work posits that small deviations ($dl \approx 0$), negligible locally, may accumulate over vast cosmic distances, creating the illusion of novel fields and forces responsible for dark phenomena.

The Weyl Integrable Geometry provides an elegant resolution to this puzzle. It suggests that constructive interference of distant phenomena at the observer's location could mimic the effects of DE and DM, thus reconciling these observations with Einstein's original objection. Crucially, non-integrable contributions in this geometry are suppressed by destructive interference, aligning the framework with local observations while simultaneously explaining large-scale anomalies.

This insight illuminates a path toward unifying our understanding of local physics and cosmic-scale phenomena, bridging the gulf between GR and the challenges posed by DE and DM. The elegance of WIG lies in its capacity to incorporate Einstein's insights while transcending the limitations of the original GR framework, reinforcing its relevance in addressing one of modern physics' most profound mysteries.

The principle of scale invariance, originally advanced by [9], gained further traction through its application to scale-covariant cosmology by [10]. These early investigations drew inspiration from the Large Numbers Hypothesis of [11], a concept that, while thought-provoking, faced considerable skepticism due to its speculative nature. More recently, the emergence of the Scale-Invariant Vacuum (SIV) cosmology in 2016 [12–14] and subsequent elaborations by [15] in 2017 have rejuvenated the discourse. Since its inception, the SIV paradigm has been rigorously tested against diverse cosmic phenomena [16–18], with comprehensive reviews highlighting its successes [19, 20]. Theoretical underpinnings were re-examined by [21], while its potential linkage to the enigmatic Dark Matter (DM) and Dark Energy (DE) was proposed in 2020 [22], and further elucidated the DE and DM phenomena within the SIV paradigm in 2025 [23].

Despite its growing body of evidence, scale-invariant cosmology has not escaped the shadow of skepticism, a remnant of early critiques. This hesitancy may partly explain the under appreciation of the SIV framework, despite the promising results highlighted in recent analyses [19]. The SIV paradigm has shown remarkable success in explaining various astrophysical phenomena, including galactic rotation curves and the growth of density fluctuations, without invoking dark matter [22, 24]. Despite its Lagrangian formulation [21], the framework lacks a dynamical origin/physics for understanding λ, instead it is relying on heuristic arguments like the scale invariance of empty space and homogeneity of space at large cosmic scales that may not be applicable at all scales and in presence of matter. To address this, a new framework—*Reparametrization Invariant Symmetry Scaling* (RISS) [25,26]—has been proposed, in which the gauge factor emerges from the principle of time reparametrization invariance of the physical laws, rather than from vacuum symmetry assumptions [23]. The RISS paradigm, while also rooted in scale invariance, takes a different approach to the fundamental nature of the conformal scale transformation and its implication for the known physical laws [27]. This leads to potentially distinct predictions for phenomena such as primordial nucleosynthesis.

In this work, I interpret DE as the Einstein cosmological constant Λ_E and DM through a MOND-like acceleration relation, $g \sim \sqrt{a_0 g_N}$. These concepts are explored within the Reparametrization Invariant Scaling Symmetry (RISS) framework, mathematically equivalent to SIV cosmology in the context of a homogeneous and isotropic Universe. Precise formulas and numerical values for Λ_E and a_0 are derived and computed under this paradigm.

The limitations of traditional MOND in explaining certain cosmological observations are well known. However, the paper does not aim at deriving MOND but merely recognizes that upon application of the reparametrization invariant scaling symmetry, which is similar to the SIV framework, one can find a MOND-like behavior from first principles rooted in reparametrization invariance. This may be viewed as a relativistic extension of MOND, addressing one of its primary criticisms, which is now not an issue given the variety of relativistic extensions to MOND like RMOND and so on. Furthermore, recent studies have explored extensions to MOND, such as incorporating sterile neutrinos, to account for phenomena like the Bullet Cluster and CMB anisotropies. In particular, the CMB anisotropies problem seems to have been resolved, see [28]. Some studies applying relativistic MOND theories, including TeVeS, to model the Bullet Cluster have claimed consistency with gravitational lensing observations [29], but these analyses are complex, and the extent to which they fully resolve the mass discrepancy is still a topic of ongoing research and debate [30]. The Bullet Cluster remains a controversial system for both MOND and ΛCDM [31,32]. While challenges remain, the current manuscript contributes to this ongoing effort by offering a novel theoretical basis for MOND-like behavior [23, 33], without the need of all the extra epicycles of dark phantoms or ad-hoc modification to the fundamental equations.

Reparametrization invariance offers a profound geometric foundation for understanding fundamental interaction fields, particularly classical long-range forces [34], while also shedding light on essential properties of physical systems [27]. In contrast to scale invariance, which often faces theoretical challenges in the presence of matter [35], RISS emerges as a symmetry as fundamental as diffeomorphism invariance. This added layer of symmetry extends the observer's toolkit for understanding natural phenomena.

Through this perspective, RISS becomes a powerful framework for addressing DE and DM, suggesting it may pave the way for a transformative approach to these mysteries. The fusion of symmetry, geometry, and observational consistency positions RISS as a potentially indispensable principle in the evolving landscape of modern cosmology.

1. Reparametrization Invariant Scaling Symmetry (RISS)

Reparametrization invariance holds a place of significance in the study of physical systems. Its importance arises from its foundational role in diverse areas of physics, yet it often introduces considerable complexity. Systems governed by this invariance typically exhibit a Hamiltonian identically equal to zero, making them challenging to analyze. Moreover, such systems are invariably associated with gauge symmetries, including the diffeomorphism invariance of General Relativity, the gauge invariance of electromagnetism, and the gauge symmetries pervasive in particle physics and string theory [27, 34].

Gauge symmetries, by their nature, do not alter the underlying dynamics of the systems or processes they describe. Analogous to the choice of coordinate systems, they leave the physics unchanged while influencing the manner of description. The judicious selection of a gauge or a coordinate system often simplifies the equations of motion and enhances our understanding of the system under study. Reparametrization invariance, in this regard, functions as a gauge symmetry, permitting freedom in the choice of process parametrization. While coordinate time or proper time are frequently employed, reparametrization invariance allows for any reasonable parameterization of time for a process under study.

At first glance, it may seem that reparametrization invariance introduces no fundamentally new physics. This assumption holds true, however, only if the equations utilized in modeling and analysis are themselves reparametrization invariant. To illustrate, consider the well-known Newtonian gravity equation:

$$\frac{d^2\overrightarrow{r}}{dt^2} = -\frac{GM}{r^2}\frac{\overrightarrow{r}}{r}. \tag{1}$$

This equation has been extensively tested and validated within the Solar System and serves as the cornerstone of Newtonian physics. Yet, its validity is conditional. Specifically, it applies when the time variable t represents the proper time of the test object under study. If a reparametrization of time is performed, such as $dt \to \lambda(t)\, dt$, the left-hand side of the equation acquires additional terms, thereby altering its form.

This alteration reveals that Eq. (1) is not reparametrization invariant. Consequently, any conclusions derived from it are restricted to the comoving frame where Newtonian physics holds true. Thus, to assert that no new physics arises from reparametrization invariance, one must ensure that the equations of motion already employ this symmetry. Failure to do so invites the possibility of "novel" phantom effects or corrections. That is, the reparametrization symmetry will induce terms that, when not taken into account, may be mistaken for new physics, while the problem is that we are not viewing the system in its true comoving frame using the proper time parametrization. This will be shown to be the case for the equation of motion due to the terms induced by the reparametrization symmetry that are easily mistaken for extra forces incorrectly interpreted as presence of a dark matter component. In a similar way the extra terms in the equations for cosmology generate the Einstein cosmological constant; that is, when we try to recover the scale invariance as attempted by Dirac and others, it can be recognized as extra kinetic energy due to relative temporal motion.

The lesson is clear: reparametrization invariance is not merely a theoretical convenience (or inconvenience depending on the view point). Instead, it serves as a critical lens through which to examine the fundamental equations governing physical systems. It safeguards the integrity of our models and ensures their applicability across diverse frames, reaffirming the centrality of symmetry in modern physics.

Another critical property to be preserved under reparametrization is the constancy of the speed of light, c. This requirement stems from the modern understanding of fundamental constants, such as c, $\hbar$, and k_B, which are regarded as conversion factors set by the choice of units [36]. These constants underpin the fabric of physical laws and must remain invariant under any transformation to maintain the coherence of theoretical models.[1]

In this context, when applying a time reparametrization $dt \to \lambda(t)\,dt$, one must simultaneously rescale spatial coordinates as $dr \to \lambda(t)\,dr$. This dual transformation ensures that the speed of light c, critical for the study of light null-geodesics, remains unaltered. The invariance of c is essential not only for preserving the mathematical structure of relativistic equations but also for maintaining the physical consistency of measurements and observations across transformed reference frames.

This necessity naturally introduces a scaling symmetry, formalized through the Weyl transformation. The transformation modifies the metric tensor as follows:

$$g_{\mu\nu} \to \lambda^2 g_{\mu\nu}, \tag{2}$$

where $\lambda(t)$ is the scaling factor. Such transformations extend the principle of invariance by incorporating the scale of the underlying spacetime geometry, allowing a unified description of systems under reparametrization.

The preservation of c during reparametrization highlights the interplay between symmetry and fundamental constants, reinforcing the robustness of the theoretical frameworks in modern physics. By incorporating the Weyl transformation, one ensures that the physical principles governing the null-geodesics of light and the invariance of fundamental constants are seamlessly integrated, a testament to the elegance and consistency of the reparametrization-invariant paradigm.

The crucial insight here is that the Weyl transformation, (2) naturally emerges from the interplay of the reparametrization gauge and the principle of preserving the constancy of the speed of light. This invariance is indispensable, as any deviation—such as allowing for concepts like tired light—introduces frameworks that obscure the fundamental symmetry at work [37, 38].

In the context of the Reparametrization Invariant Scaling Symmetry (RISS) framework, the scaling factor λ is intrinsically assumed to depend solely on time. This assumption simplifies the analysis, as λ becomes a function of t only, consistent with the symmetry requirements of scale-invariant cosmology in a homogeneous and isotropic universe [21, 22]. In contrast, the general Weyl transformation allows λ

[1] A general time reparametrization can easily justify models with varying speed of light; thus, one can easily understand the success of such models in refitting various other model parameters. However, upon reflection on the modern choice if units one recognizes that a conformal transformation is a more reasonable choice (2); thus, keeping the constancy of the speed of light c. In a similar way the choice of how to define the units of energy results in specific choices, i.e. of keeping fixed either $\hbar$, G, or a specific mass, say m_e. The choice to be made will depend on the process to be studied. For example, for cosmology keeping G fixed will imply specific time variation of $\hbar$ and the rest mass unit. This is the SIV gauge where Λ_E is also a constant and $\hbar$ is practically zero, however, when considering the BBN the choice of fixed $\hbar$ is better and does show a promising resolution of the Li-7 problem.

to depend on all spacetime coordinates. The restriction to a time-only dependence in RISS reflects the constraints of cosmological symmetry and reinforces the compatibility with the isotropic and homogeneous structure of the Universe.

At this juncture, as long as the equations of motion respect reparametrization invariance, the specific functional form of $\lambda(t)$ remains undetermined. However, one can glean qualitative insights about $\lambda(t)$ and its associated terms within the framework of the equations of motion. These preliminary understandings set the stage for further exploration in subsequent sections.

To derive a precise functional form for $\lambda(t)$, one must analyze a specific system and its defining equations. This endeavor will be addressed in the forthcoming discussions on scale-invariant cosmology, with a particular focus on the Scale-Invariant Vacuum (SIV) cosmology. Here, the connection between symmetry, scaling, and the observable Universe will take on a more concrete form, offering a deeper understanding of how $\lambda(t)$ manifests in the dynamics of cosmic evolution.

1.1. RISS effects on the Equations of Motion

Consider the left-hand side (LHS) of Eq. (1), transformed under the reparametrizations $dt \to \lambda(t)\,dt$ and $dr \to \lambda(t)\,dr$, induced by the Weyl transformation (2):

$$\frac{1}{\lambda}\frac{d}{dt}\left(\frac{1}{\lambda}\frac{d}{dt}(\lambda\overrightarrow{r})\right) = \frac{1}{\lambda}\frac{d^2\overrightarrow{r}}{dt^2} + \frac{\dot{\lambda}}{\lambda^2}\frac{d\overrightarrow{r}}{dt} + \frac{\overrightarrow{r}}{\lambda}\frac{d}{dt}\left(\frac{\dot{\lambda}}{\lambda}\right). \tag{3}$$

This transformation modifies the Newtonian dynamics, as (1) becomes:

$$\frac{d^2\overrightarrow{r}}{dt^2} = -\left(\frac{GM}{r}\right)\frac{\overrightarrow{r}}{r^2} + \kappa\,\frac{d\overrightarrow{r}}{dt} + \left(\frac{d\kappa}{dt}\right)\overrightarrow{r}, \tag{4}$$

where $\kappa = -\dot{\lambda}/\lambda$.

In general, if the κ-related terms were already present on the right-hand side of Eq. (1), then rescaling the time and space variables would induce the transformation $\kappa \to \kappa - \dot{\lambda}/\lambda$.

Interpretation and Gauge Choice: *The transformation reveals a profound symmetry: it is possible to choose $\lambda(t)$ such that κ becomes zero. This choice corresponds to the* **proper time parametrization***, in which the equations of motion return to their familiar laboratory form.*

In this context, the scale factor $\lambda(t)$ reflects the deviation of the chosen time parametrization from proper time. When $\lambda(t)$ is appropriately calibrated, the extraneous terms introduced by reparametrization vanish, simplifying the dynamics. This gauge choice is not merely a mathematical convenience; it restores the canonical form of the equations of motion, ensuring they align with the fundamental principles observed in laboratory and local astrophysical contexts.

The role of $\lambda(t)$ is thus pivotal—it encodes the adjustments necessary to reconcile time reparametrization with physical dynamics. By understanding its behavior, one gains insight into the nature of deviations from proper time and the underlying symmetries governing the system. This perspective emphasizes the interplay between reparametrization invariance and the foundational principles of mechanics, unifying the description across different frames and parametrizations.

1.2. RISS as framework for scale invariant cosmology

As previously articulated, the application of Weyl geometry as a foundational framework for scale-invariant cosmology has been rigorously examined by [9, 10]. Furthermore, studies such as [39, 40] have extended these considerations into the realm of potential astronomical applications. More recent investigations, however, have turned toward profoundly abstract mathematical formulations [41].

It must be emphasized that *nature-oriented explorations of phenomena capable of observational validation remain notably sparse.* Only a handful of examples exist in this domain [19, 40, 42]. By contrast, certain misguided approaches, such as those invoking a varying speed of light [37, 38], are inconsistent with the foundational principles upon which our contemporary International System of Units is constructed [36].

Moreover, applications of this framework within the realm of particle physics have received minimal attention, with rare exceptions such as [43]. Here, I aim to underscore the essential characteristics of the Ricci tensor and scalar, both of which are pivotal for contextualizing the present theoretical perspective. These insights will later serve to elucidate a specific form of $\lambda(t)$ that can be aligned with the SIV gauge choice.

The central role of a time-dependent λ within the RISS context under the Weyl transformation (2), and its implications for the corresponding equations of motion, has been previously addressed. Formal discussions on this subject have been framed within the broader context of Dirac's co-calculus and its applications [19, 21]. In this work, I limit the discussion to the minimal formalism necessary to substantiate the $\lambda(t)$ dependence and its connection to the SIV gauge.

Upon subjecting the metric to a Weyl transformation [39,40], the resulting expressions for the Ricci tensor and scalar emerge as critical elements of analysis:

$$g_{\mu\nu} \to \lambda^2 g_{\mu\nu} \Rightarrow R_{\mu\nu} \to R_{\mu\nu} + K_{\mu\nu}, \tag{5}$$

where $K_{\mu\nu}$ is given in terms of $\kappa_\mu = -\partial_\mu \ln\lambda$ as:

$$K_{\mu\nu} = g_{\mu\nu}\kappa^\rho\kappa_\rho + 2\kappa_\mu\kappa_\nu + \kappa_{\mu;\nu} + \kappa_{\nu;\mu} - 2g_{\mu\nu}\kappa^\rho_{;\rho}. \tag{6}$$

Such expressions have been discussed in Eq. (89.2) by [8] first and later by [9,43] as well. After contracting (5) one obtains the Ricci scalar to be mapped so that $R \to (R+K)/\lambda^2$ along with:

$$K = 6\kappa^\rho\kappa_\rho - 6\kappa^\rho_{;\rho}. \tag{7}$$

In the general framework of Weyl geometry, λ is allowed to vary across both space and time, thereby imparting corresponding variability to the associated connexion vector κ_μ. Here, I adopt the French term “connexion"to avoid conflating this concept with the standard “connection"employed in Einsteinian General Relativity (EGR).

2. The SIV gauge for $\lambda(t)$

The Einstein field equations in the presence of a cosmological constant, Λ_E, take the form:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_E g_{\mu\nu} = \varkappa T_{\mu\nu}. \tag{8}$$

This formulation is rooted in the assumption of a metric-compatible connection, where the Einstein tensor satisfies the zero-divergence condition, $\nabla^\mu G_{\mu\nu} = 0$, with $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. Likewise, the stress-energy tensor, $T_{\mu\nu}$, obeys the covariant conservation law $\nabla^\mu T_{\mu\nu} = 0$, ensuring consistency across the framework. Provided $\varkappa = 8\pi G/c^4$ and Λ_E are constants, the equations remain self-consistent. Deviations, however, lead to the relation $\Lambda_{E,\mu} = \varkappa_{,\nu} T^\nu_\mu$.

The left-hand side (LHS) of (8) encapsulates the gravitational influence via the metric tensor and its connection, while the right-hand side (RHS) represents matter and its energy density. The constancy of Λ_E emerges naturally from the constancy of G and c.

It has been proposed that the term $\Lambda_E g_{\mu\nu}$ could be relocated from the LHS to the RHS of (8) to associate Λ_E with the vacuum energy or zero-point energy. Yet, this reallocation precipitates the **cosmological constant problem**, which is marked by a staggering disparity between quantum field theory (QFT)-based predictions and observationally inferred values [1, 44–46].

Introducing a constant energy density linked to non-gravitational fields in a homogeneous and isotropic universe further suggests the presence of a black hole event horizon at a sufficiently large distance, R_S. This premise posits that the Universe might exist within a black hole, as speculated in [47, 48]. However, such a scenario implies a contracting flow of matter toward a central region with maximal density, whereas observations indicate an expanding Hubble flow directed toward the future event horizon.

An alternative perspective arises by considering the observed age of the Universe, $\tau_0 \approx 1/H_0$, along with c and G, which permits an accurate estimation of Λ_E. Within this paradigm, Λ_E attains a novel interpretation. To investigate further, one applies the Weyl transformation (2), which relates the EGR-frame metric, $g'_{\mu\nu}$, to the WIG-frame metric, $g_{\mu\nu}$, via a factor λ:

$$g'_{\mu\nu} = \lambda^2 g_{\mu\nu}.$$

Here, primed quantities correspond to the EGR frame, and unprimed quantities refer to the generalized WIG frame. Substituting equations (5), (6), and (7) into the Einstein equation (8), the Weyl transformation reformulates the Einstein equation in the WIG framework:

$$R_{\mu\nu} + K_{\mu\nu} - \frac{1}{2}\lambda^{-2}(R+K)\lambda^2 g_{\mu\nu} + \Lambda_E \lambda^2 g_{\mu\nu} = \varkappa T_{\mu\nu}. \tag{9}$$

Within the paradigm of Reparametrization Invariant Scaling Symmetry (RISS), the local time coordinate does not uniquely dictate the evolution of the Universe. This flexibility allows for a decomposition of equation (9) into two separate equations:

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \varkappa T_{\mu\nu} - \tilde{T}_{\mu\nu}, \tag{10}$$

$$K_{\mu\nu} - \frac{1}{2}K g_{\mu\nu} + \Lambda g_{\mu\nu} = \tilde{T}_{\mu\nu} \approx 0. \tag{11}$$

The separation of (9) reflects a judicious choice of gauge for λ when the metric $g_{\mu\nu}$ is conformally equivalent to Minkowski metric[2]. The second equation, (11), governs λ and ensures that (10) is left independent of λ. Taking the trace of (11), one finds $\Lambda = K/4$. With λ being time-dependent, as required for reparameterization ($d\tau' = \lambda\, dt$), and using $\kappa = \kappa_0 = -\dot{\lambda}/\lambda$, the SIV equations are derived:

$$\Lambda = \frac{3}{2}\left(\kappa^2 - \dot{\kappa}\right) \Rightarrow \Lambda = 3\kappa^2 = 3\left(\frac{\dot{\lambda}}{\lambda}\right)^2 \text{ if } \dot{\kappa} = -\kappa^2. \tag{12}$$

In cosmology, the assumption of a homogeneous and isotropic Universe typically restricts λ to pure time-dependence. However, within the RISS framework, this time dependence of λ is fundamental and does not preclude the Universe from attaining inhomogeneous or anisotropic states. *Thus, the question of whether scale invariance represents an intrinsic symmetry of nature is inconsequential within the RISS paradigm.*

By taking the time derivative of κ/λ, it follows that:

$$\frac{d}{dt}\left(\frac{\kappa}{\lambda}\right) = \frac{\dot{\kappa} + \kappa^2}{\lambda},$$

which vanishes if $\dot{\kappa} = -\kappa^2$. Under this condition, Λ/λ^2 remains a constant, which one denotes as Λ_E, recognizing that these two constants will ultimately coincide. Consequently, in the case where $\dot{\kappa} = -\kappa^2$, the solution for $\lambda(t)$ is straightforward and can be characterized using the constant $\Lambda_E = \Lambda/\lambda^2 = 3\dot{\lambda}^2/\lambda^4$:

$$\varepsilon(t_0 - t)\sqrt{\frac{\Lambda_E}{3}} = \frac{1}{\lambda} - \frac{1}{\lambda_0}. \tag{13}$$

This implies:

$$\kappa^2 = \frac{\dot{\lambda}^2}{\lambda^2} = \frac{\Lambda}{3} = \left(\frac{\Lambda_E}{3}\right)\lambda^2,$$

[2] If the metric $g_{\mu\nu}$ is not conformally equivalent to Minkowski metric, the expression for λ to be introduced later, can be used in (11) to define an effective dark energy term $\tilde{T}_{\mu\nu}$ that will enter the RHS of (10) with a negative sign.

along with the solution:

$$\lambda = \frac{\lambda_0}{1 + \lambda_0 \varepsilon (t_0 - t)\sqrt{\frac{\Lambda_E}{3}}}, \tag{14}$$

and the corresponding expression for κ:

$$\kappa = -\frac{\dot{\lambda}}{\lambda} = -\varepsilon\lambda\sqrt{\frac{\Lambda_E}{3}}.$$

These relationships form the cornerstone of the new paradigm. Furthermore, from $\dot{\kappa} \sim \dot{\lambda} = \varepsilon\lambda^2\sqrt{\Lambda_E/3}$, it follows that:

$$\dot{\kappa} = -\varepsilon^2 \left(\frac{\Lambda_E}{3}\right)\lambda^2 = -\kappa^2,$$

as required. Here, $\varepsilon = \pm 1$, ensuring that $\varepsilon^2 = 1$.

By setting $\lambda_0\sqrt{\Lambda_E/3} = 1/t_0$, one obtains:

$$\lambda = \frac{\lambda_0 t_0}{t},$$

for $\varepsilon = -1$. Consequently, $\kappa = -\dot{\lambda}/\lambda = 1/t$ with $t \in [t_{\text{in}}, t_0]$, where t_{in} corresponds to the Big Bang moment, when $a(t_{\text{in}}) = 0$ [22].

While the mathematical transformation itself is straightforward, its physical implications are profound, and the way to arrive at such a transformation is not trivial but mainly guided by the principle of reparametrization invariance of the physical laws. In order to arrive at a logarithmic time parameterization, a specific choice of $\lambda = dt'/dt$ has to be made, but this requires one to follow the Dirac idea for scale-invariant cosmological equations, that is easily justified within the paradigm of reparametrization invariance. The ultimate conclusion is that the observed acceleration of the Universe is a manifestation of un-proper time parameterization rather than a manifestation of an unknown dark-energy component.

It is often convenient to set $\lambda_0 = 1$ and adopt SIV time units such that $t_0 = 1$. This choice leads to the same expression for the scale factor λ as derived under the fundamental SIV hypothesis, which posits that macroscopic empty space is scale-invariant, homogeneous, and isotropic [15].

Within the RISS paradigm, however, the time dependence of λ is a foundational assumption. This does not preclude the Universe from existing in anisotropic or inhomogeneous states. Thus, the question of whether scale invariance constitutes a true symmetry of nature is irrelevant within the RISS framework.

The flexibility of the RISS framework also allows for parametrizations where $\dot{\kappa} = -\kappa^2$. This choice yields $\kappa^2 = \Lambda/3$ and ensures that $\Lambda_E = \Lambda/\lambda^2$ remains a constant. Such requirement for $\lambda(t)$ seems to be essential for eliminating phantom pressure terms that has no energy counterpart component in the relevant FLRW framework extension.

3. Interpretation of the Cosmological Constant within the New Framework

The inclusion of a non-zero cosmological term, Λ_E, raises profound questions. Foremost among these are: (1) the discordance between the observed value of Λ_E and theoretical predictions based on quantum field theory (QFT) zero-point energy[3], and (2) the curious implication that the Universe could be interpreted as existing within a black hole, if the constant energy density of the vacuum was due to actual physical particles and fields.

The present paradigm offers a resolution to these challenges by demonstrating that Λ_E must be a constant. As previously discussed, the constancy of the Einstein Cosmological Constant, Λ_E, is a direct

[3]The zero-point energy explanation often focuses on the non-zero ground state of the harmonic oscillator and overlooks the energy of the source for this harmonic oscillator potential.

consequence of assuming the Newton Gravitational Constant, G is a true constant. This insight permits the selection of an alternate parametrization that eliminates the cosmological term, Λ_E.

The approach presented is not merely an absorption of Λ_E into a time redefinition; it offers a reinterpretation of Λ_E as related to a constant extra energy density due to relative motion through the temporal dimension of the spacetime continuum. Thus, this manifestation of DE, interpreted through the framework of RISS, Λ_E emerges as a "kinetic energy" term, derived from relative temporal motion; one has to set it apart from the conventional spatial motion-based kinetic energy. By doing so, the paper provides a resolution to the cosmological constant problem by showing that the small observed value of Λ_E is naturally related to the age of the Universe. Thus, it has an anthropic explanation and does not need any extra fields or particles. This way, the value of Λ_E matches the observed value; therefore, the problem is resolved. Addressing the cosmological constant problem that way, may not be what the current common wisdom expects; however, one has to accept that maybe nature does not use any additional dark entities. After all, we have not detected such entities in any laboratories for the past 25 years and even more.

This approach suggests that the additional energy density attributed to the cosmological constant is an observer-dependent effect. Analogous to how the kinetic energy of a system varies with the relative motion between systems, here the effect is tied to discrepancies in time parametrization.

To elucidate, consider that the metric $g_{\mu\nu}$ provides the Λ-free Einstein General Relativity (EGR) equations, as given in (10). The appearance of the constant Λ_E arises solely from the choice of a conformal EGR metric tensor:

$$g'_{\mu\nu} = \lambda^2 g_{\mu\nu},$$

where the scaling factor $\lambda = t_0/t$ is defined by $\sqrt{\Lambda_E/3} = 1/t_0$, as derived in (14). This connects the cosmological constant, Λ_E, to the age of the Universe, t_0, aligning with dimensional considerations [49].

Expressing these relations in SI units, where the age of the Universe is approximately $\tau_0 = 13.8$ billion years, and the speed of light $c = 3 \times 10^8$ m/s, one calculates:

$$\Lambda_E = \frac{3}{(c\tau_0)^2} \approx 1.8 \times 10^{-52}\,\mathrm{m}^{-2}, \tag{15}$$

a value consistent with observational estimates [1, 49]. Notably, this measured value is derived through model fitting within the ΛCDM framework.

Within this framework, the time dependence of λ reveals that, with a suitable choice of parametrization, the Λ_E term vanishes when described using an equivalent conformal metric $g_{\mu\nu}$. This leaves only (10), free from the observer-dependent cosmological constant Λ_E expressed in (15).

Thus, the cosmological constant, Λ_E, can be reinterpreted as an energy density arising from the choice of an improper time parametrization for the Universe. Specifically, the additional energy density results from the observer's relative motion through time, analogous to how a non-zero kinetic energy arises when the observer is not in a co-moving frame. This interpretation provides a conceptual parallel to relativity's treatment of energy as frame-dependent, but here extends the principle to the domain of cosmological time.

4. Deriving the MOND fundamental relation and the value of a_0

The SIV paradigm, being rooted in the principle of scale invariance, compels a reexamination of the equations of motion in a cosmological framework. The derivation of the scale factor $\lambda(t)$ within this context, for a homogeneous and isotropic Universe, aligns with the RISS principle where λ depends solely on time. This adherence to scale invariance necessitates a modification of the classical equations of General Relativity (GR). Specifically, the adoption of Dirac's co-calculus enables the inclusion of an additional velocity-dependent term, yielding a more generalized equation of motion. In the simplified cases of both SIV and RISS frameworks, the co-calculus reduces to a single nonzero component, κ_0. Consequently, the equations of motion take the form:

$$\frac{d^2\overrightarrow{r}}{dt^2} = -\left(\frac{G_t M(t)}{r}\right)\frac{\overrightarrow{r}}{r^2} + \kappa(t)\frac{d\overrightarrow{r}}{dt}, \tag{16}$$

where $\kappa(t) = -\frac{\dot{\lambda}}{\lambda}$. Within the SIV gauge $\lambda(t) = \frac{t_0}{t}$, the term $\dot{\kappa}\overrightarrow{r}$, which appears in (3), can be neglected in favor of the dominant $\kappa\overrightarrow{v}$ term. This simplification follows from the relationship $\dot{\kappa} = -\kappa^2$, which introduces corrections on the order of $1/\tau_0^2$, and thus becomes negligible for cosmologically relevant timescales.

A key insight is that for the term $\frac{G_t M(t)}{r}$ to exhibit scale invariance, the product $G_t M(t)$ must scale in proportion to r. Within the RISS framework, this necessitates a rescaling of G_t to preserve invariance under reparameterizations while fixing units, ensuring that energy and mass, scale oppositely to time. That is, keeping $\hbar$ fixed instead of G or some choice of mass unit such as m_e. In contrast, under the SIV paradigm, G_t is treated as a true constant, demanding a time dependence for the mass $M(t)$. This scale invariance can be expressed as $G_t M(t) \propto \lambda$, and with G_t held constant, one derives $M(t) = M_0\lambda(t)$. This variation of mass with time is consistent with the conservation laws embedded in the scale-invariant formalism [12, 15]. For simplicity, one will henceforth suppress the explicit time dependence of G_t and $M(t)$ in the forthcoming notation.

4.1. Velocity-Dependent Term and Symmetry Considerations

The velocity-dependent term in (16) has been rigorously derived in the weak-field limit of the SIV framework [21, 39, 40, 50], assuming the spacetime metric $g_{ij} = -\delta_{ij}$, $g_{00} = 1 + \frac{\Phi}{c^2}$, with $\Phi = -\frac{GM}{r}$ representing the Newtonian potential. This derivation employs $\Gamma^i_{00} = \frac{1}{c^2}\partial^i\Phi$, where $\partial^i = g^{i\alpha}\frac{\partial}{\partial x^\alpha}$. By contrast, in the RISS paradigm, the $\kappa(t)\vec{v}$ term emerges from reparameterization invariance without the need for a weak-field approximation [34].

The insistence on reparameterization symmetry imposes the necessity of the κ-term to account for the dynamics under transformations of the time parameter. Crucially, in a proper time parameterization, κ naturally vanishes. However, relying on an observer's coordinate time introduces discrepancies that correspond to an improper parameterization, thereby necessitating the inclusion of κ to restore the broken symmetry.

This consideration underscores a profound truth: the absence of reparameterization symmetry in the modeling of physical systems leads to an incomplete representation. The κ-term, arising as a fictitious velocity-dependent acceleration, serves to restore this missing symmetry [34], whether it be reparameterization invariance or scale invariance, depending on the specific formalism. Thus, the $\kappa\vec{v}$ term is indispensable in scenarios where the system is described using coordinate time rather than proper time.

4.2. Distinction Between SIV and RISS Approaches

In the SIV framework, the scale factor $\lambda(t)$ is typically expressed as $\frac{t_0}{t}$, a choice justified by the symmetry constraints of the paradigm. However, reparameterization symmetry, as realized in the RISS framework, allows for an arbitrary time dependence of $\lambda(t)$ [34]. This difference in perspective underscores the broader significance of the $\kappa\vec{v}$ term: its inclusion restores the symmetry that would otherwise be broken when coordinate time is misaligned with the proper time of the system.

Thus, the presence or absence of the $\kappa\vec{v}$ term serves as a diagnostic tool for identifying symmetry violations in the underlying mathematical formulation. Proper modeling demands its inclusion to maintain the consistency and completeness of the theory.

4.3. Deriving the MOND-like fundamental relation

To establish an expression for the MOND acceleration a_0, one begins by contemplating the ratio of the Newtonian acceleration, $g_N = GM/r^2$, to the supplementary dynamic acceleration, $\kappa(t)v$, as expressed through their magnitudes:

$$x = \frac{\kappa v r^2}{GM}. \tag{17}$$

Invoking the instantaneous radial acceleration relation $v^2/r = GM/r^2$, one eliminates the velocity v. Then, substituting $g_N = GM/r^2$, the expression for x refines to:

$$x = \frac{\kappa v r^2}{GM} = \kappa\sqrt{\frac{r^3}{GM}} = \kappa\sqrt{\frac{r}{g_N}}.$$

The parameter x, explored extensively in earlier studies [17, 18, 50, 51], emerges as pivotal. Its role in unifying the SIV framework with MOND, as elaborated in [51], underscores its significance. Here, our emphasis lies on deriving precise formulations while explicitly incorporating the temporal dimension $\kappa = \kappa_0 = -\dot{\lambda}/\lambda$, contextualized within the WIG paradigm.

When the dynamic acceleration dominates over the Newtonian counterpart ($x \gg 1$), the governing relation simplifies:

$$g = g_N + x g_N \approx x g_N = \kappa\sqrt{r g_N}.$$

This leads directly to the MOND-like relation $g \sim \sqrt{a_0 g_N}$, from which a_0 is inferred as:

$$a_0 \approx \kappa^2 r.$$

Considering $r \to r_H = c/H_0$, one derives:

$$a_0 \approx \kappa^2 r_H = \kappa^2 c/H_0. \tag{18}$$

4.4. On the value of the MOND-like acceleration a_0.

In natural units where $c = \lambda_0 = t_0 = 1$ (with c being the speed of light), the time component of the connexion vector reduces to $\kappa = 1/t$. During the matter-dominated epoch, employing the SIV scale factor $a(t) = \left((t^3 - \Omega_m)/(1 - \Omega_m)\right)^{2/3}$ [52], the Hubble parameter becomes:

$$H = \frac{2t^2}{t^3 - \Omega_m}.$$

Thus:

$$a_0 \approx \frac{\kappa^2}{H_0} = \frac{1 - \Omega_m}{2} = \frac{\Omega_\lambda}{2},$$

where in the SIV framework, $\Omega_\lambda = 2/(Ht)$, assuming spatial flatness ($\Omega_k = 0$). Accordingly, the Friedmann equation, $\Omega_\lambda + \Omega_m = 1$, holds trivially within SIV. While extensions to non-zero curvature models ($k \neq 0$) are feasible [15], our focus remains on the flat case. Hence, the presence of a non-zero κ-term in equation (16) inherently generates a non-zero MOND acceleration, as expressed in (18).

To express the MOND acceleration a_0 in conventional SI units, one begins by noting that the age of the Universe is $\tau_0 = 13.8$ billion years, the Hubble constant is $H_0 = 68\,\mathrm{km/s/Mpc}$, and the speed of light is $c = 3 \times 10^8\,\mathrm{m/s}$. Using the approximation $\kappa \approx 1/\tau_0$, a preliminary estimate yields:

$$a_0 \approx \frac{c}{\tau_0},$$

given that $H_0\tau_0 \approx 1$. However, a more precise determination requires incorporating the full relationship between κ and the scale factor λ, where $\kappa = -\dot{\lambda}/\lambda$. In this framework, $\kappa(\tau) = (dt/d\tau)\kappa(t)$.

To compute $dt/d\tau$, consider:

$$\frac{t - t_{\mathrm{in}}}{t_0 - t_{\mathrm{in}}} = \frac{\tau - \tau_{\mathrm{in}}}{\tau_0 - \tau_{\mathrm{in}}},$$

for $\tau_{\mathrm{in}} = 0$ and $t_{\mathrm{in}} = \Omega_{\mathrm{m}}^{1/3}$, yielding:

$$t = \Omega_{\mathrm{m}}^{1/3} + \frac{\tau}{\tau_0}(1 - \Omega_{\mathrm{m}}^{1/3}),$$

and consequently:

$$\frac{dt}{d\tau} = \frac{t_0 - t_{\rm in}}{\tau_0} = \frac{1 - \Omega_{\rm m}^{1/3}}{\tau_0}.$$

This introduces a correction factor to the approximation $\kappa \approx 1/\tau_0$, leading to:

$$\kappa(\tau_0) = \frac{1 - \Omega_{\rm m}^{1/3}}{\tau_0}.$$

Substituting into the expression for a_0, one obtains:

$$a_0 \approx (1 - \Omega_{\rm m}^{1/3})^2 \frac{c}{\tau_0} = (1 - \Omega_{\rm m}^{1/3})^2 \frac{cH_0}{\xi},$$

where $H_0\tau_0 = \xi \approx 1$.

For a baryonic matter density parameter $\Omega_m = \Omega_b = 5\%$, this yields an estimated MOND acceleration:

$$a_0 \approx 2.75 \times 10^{-10}\,\mathrm{m/s^2}.$$

This value aligns with the predictions of the BBN and the ΛCDM model, where $\Omega_b \approx 5\%$ represents baryonic matter.

Within the SIV framework, the parameter estimates are still in progress. An alternative approach derives Ω_m from the self-consistency condition $H_0\tau_0 = \xi \approx 1$. Solving the SIV equation:

$$\frac{2(1 - \Omega_m^{1/3})}{1 - \Omega_m} \approx 1,$$

yields $\Omega_m \approx 23.6\%$. This corresponds to:

$$a_0 \approx 10^{-10}\,\mathrm{m/s^2}.$$

Both estimates of a_0 corespond to the same order of magnitude, $a_0 \sim 10^{-10}\,\mathrm{m/s^2}$, with the SIV-derived matter density $\Omega_m \approx 23.6\%$ closely paralleling the total matter content inferred in the ΛCDM model.

These results suggest that the MOND acceleration a_0 may vary with cosmological epoch, depending on the redshift of the observed system. Such potential redshift dependence could serve as a powerful discriminator, enabling rigorous tests of the predictions from both the SIV and ΛCDM paradigms [53].

Conclusion

In summary, this paper establishes that the Einstein cosmological constant, Λ_E, is a true constant within the framework presented. The scale factor $\lambda(t)$, derived from (11), is expressed as in (14), while the metric tensor satisfies the Einstein equation devoid of a cosmological constant, as shown in (10).

This approach circumvents the conceptual challenges associated with interpreting Λ_E as vacuum zero-point energy, often referred to as dark energy. Specifically, it avoids two notable puzzles: the implication that the Universe exists within a black hole and the vast discrepancy of 123 orders of magnitude between the observed value of Λ_E and predictions based on quantum field theory (QFT). Within this framework, the non-zero Λ_E of the ΛCDM model emerges from the choice of time parametrization. Much like the kinetic energy associated with spatial motion, differences in temporal parametrization introduce an “extra energy"linked to relative temporal motion.

The paper also derives an expression for the fundamental MOND acceleration, a_0, as given in (18). This acceleration governs the transition to the deep MOND regime, characterized by expected scale invariance. In this framework, the dark matter problem is addressed through the MOND paradigm, where a_0 emerges from the reparametrization invariance of the equations of motion (16). Traditional MOND limits, however, are determined by the parameter x in (17).

Current estimates place the fundamental MOND acceleration, a_0, at approximately $10^{-10}\,\mathrm{m/s^2}$, and the Einstein cosmological constant, Λ_E, at roughly $10^{-52}\,\mathrm{m^{-2}}$. Both values are consistent with their respective observational magnitudes. Notably, both constants are influenced by the age of the Universe, with a_0 also reflecting the Universe's matter content.

By deriving MOND-like behavior and reinterpreting the cosmological constant within the RISS paradigm, the present work offers a cohesive framework that addresses multiple cosmological puzzles. This approach not only provides theoretical insights, but also suggests new avenues for observational tests, and improved modeling methods. It can advance the understanding of the universe's fundamental nature by providing an explicit physics interpretation of the conformal factor $\lambda(t)$, which was not present within the SIV theory [23].

This two connections to the current cosmological timescale provide a compelling link between the evolution of the Universe and its governing dynamical principles, presenting a unified perspective on two of modern cosmology's most enigmatic constants.

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