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УРАВНЕНИЕ ДИРАКА НА ФОНЕ АКСИАЛЬНО-СИММЕТРИЧНОЙ ОБОБЩЕННОЙ МЕТРИКИ НЬЮМЕНА-УНТИ-ТАМБУРИНО

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Рассматривается множество метрик, полученное как обобщение пространства-времени Ньюмена-Унти-Тамбурино (НУТ), и исследуется безмассовая частица со спином 1/2 на фоне обобщенного пространства-времени. В рамках тетрадного формализма получено уравнение Дирака на фоне обобщенного НУТ-пространства и проведено разделение переменных. Показано, что угловые уравнения аналогичны уравнениям для исходной НУТ-метрики. Радиальная система сводится к одному дифференциальному уравнению второго порядка, и допускает решение в терминах функций Гойна при специальном выборе обобщающей функции; при таком выборе, решения радиальных уравнений получены в терминах конфлюэнтных функций Гойна.

Ключевые слова: уравнение Дирака, обобщенная метрика Ньюмена-Унти-Тамбурино, частица со спином 1/2, безмассовый случай, уравнение Гойна.

DIRAC EQUATION ON THE AXIALLY SYMMETRIC GENERALIZED NEWMAN-UNTI-TAMBURINO SPACETIME

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We consider a set of the metrics derived as a generalization of Newman-Unti-Tamburino (NUT) spacetime and study the spin 1/2 massless particle on the background of the spacetime. In the framework of tetrad formalism, the Dirac equation has been derived and the variable separation has been performed. It has been shown that the angular equations preserve the form of the equations for the original NUT metric. The radial problem has been reduced to the one differential equation of second order. The special generalized NUT metric has been chosen in such a way that the radial system has been solved in terms of confluent Heun functions.

Keywords: Dirac equation, generalized Newman-Unti-Tamburino space, spin 1/2 particle, massless case, Heun equation.

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Introduction

The aially symmetric metrics of the Taub-NUT type (so-called Euclidean Taub-NUT metrics) were actively studied within grand unified theories as a monopole-like solutions [1]. In recent years, the original NUT metric acquires a fresh interest because it has been shown that the black holes with NUT charge can be considered as physically meaningful systems. In [2], the NUT charge effects has been assessed to be tested in the spectra of quasars, supernovae, or active galactic nuclei.

The quantum-mechanical problem of particle in the background of black hole metrics is one of the interesting problems for cosmology and quantum gravity. The Dirac equation has been studied on the Kerr and Kerr-Newman spacetimes [3, 4]. It should be noted that solutions of quantum-mechanical problems in the background of black hole spacetimes have been obtained in terms of confluent Heun function [3–5], this form is convenient as it allows to study the resonant frequencies (quasispectrum), the Hawking radiation and the scattering process of scalar waves [5]. In [6], we found solutions of the Dirac equation for the massless particle in NUT background. The generalization of the Euclidean Taub-NUT metric has been proposed by Iwai and Katayama in [7, 8]. They have demonstrated that the generalization accomplishes non-trivial properties.

In this study we generalize the NUT-metric preserving its angular symmetry, study the Dirac equation in such axially symmetric background and derive the type of generalization that admits the radial solutions for massless particle in terms of confluent Heun functions.

1. Dirac equation in generalized NUT space-time

Let consider some generalization of the original NUT metric determined by the following line element

$$ds^2 = \Phi \left(dt - 2a \sin^2 \frac{\theta}{2} d\phi \right)^2 - \frac{dr^2}{\Phi} - \frac{\rho^2}{\Omega(r)} \left(d\theta^2 + \sin^2 \theta d\phi^2 \right), \tag{1}$$

$$\Phi = \frac{\Delta}{\rho^2}, \ \rho^2 = a^2 + r^2, \ \Delta = r^2 - 2Mr - a^2,$$

$\Omega(r)$ is an arbitrary function of radial coordinate r.

The metric tensor is

$$g_{\alpha\beta} = \begin{vmatrix} \Phi & 0 & 0 & 2a\Phi(1-\cos\theta) \\ 0 & -\frac{1}{\Phi} & 0 & 0 \\ 0 & 0 & -\frac{r^2+a^2}{\Omega} & 0 \\ 2a\Phi(1-\cos\theta) & 0 & 0 & 16a^2\Phi\sin^4\frac{\theta}{2} - \frac{(a^2+r^2)\sin^2\theta}{\Omega} \end{vmatrix}. \tag{2}$$

We chose the tetrad

$$e_{(a)\alpha}(x) = \begin{vmatrix} \sqrt{\Phi} & 0 & 0 & 4a\sqrt{\Phi}\sin^2\frac{\theta}{2} \\ 0 & \frac{1}{\sqrt{\Phi}} & 0 & 0 \\ 0 & 0 & \frac{\rho}{\sqrt{\Omega}} & 0 \\ 0 & 0 & 0 & \frac{\rho}{\sqrt{\Omega}}\sin\theta \end{vmatrix} \tag{3}$$

so that the relations

$$g_{\alpha\beta} = \eta^{ab} e_{(a)\alpha} e_{(b)\beta}$$

hold true.

The tetrad with upper coordinate index is given as

$$e_{(a)}^{\ \ \beta} = \begin{pmatrix} \frac{1}{\sqrt{\Phi}} & 0 & 0 & 0 \\ 0 & -\sqrt{\Phi} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{\Omega}}{\rho} & 0 \\ \frac{2a\sqrt{\Omega}}{\rho}\tan\frac{\theta}{2} & 0 & 0 & -\frac{\sqrt{\Omega}}{\rho\sin\theta} \end{pmatrix}. \tag{4}$$

In order to calculate the Ricci rotation coefficients, we apply the known formulas [9]

$$\gamma_{abc} = \frac{1}{2}(\lambda_{abc} + \lambda_{bca} - \lambda_{cab}), \quad \lambda_{abc} = (\frac{\partial e_{(a)\alpha}}{\partial x^\beta} - \frac{\partial e_{(a)\beta}}{\partial x^\alpha}) e^\alpha_{(b)} e^\beta_{(c)}. \tag{5}$$

The Ricci rotation coefficients have the following explicit form

$$\gamma_{ab0} = \begin{vmatrix} 0 & \frac{\Phi'}{2\sqrt{\Phi}} & 0 & 0 \\ -\frac{\Phi'}{2\sqrt{\Phi}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{a\sqrt{\Phi}\Omega}{\rho^2} \\ 0 & 0 & -\frac{a\sqrt{\Phi}\Omega}{\rho^2} & 0 \end{vmatrix}, \qquad \gamma_{ab1} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\gamma_{ab2} = \begin{vmatrix} 0 & 0 & 0 & \frac{a\sqrt{\Phi}\Omega}{\rho^2} \\ 0 & 0 & \frac{\sqrt{\Phi}}{2}\left(\frac{2r}{\rho^2} - \frac{\Omega'}{\Omega}\right) & 0 \\ 0 & -\frac{\sqrt{\Phi}}{2}\left(\frac{2r}{\rho^2} - \frac{\Omega'}{\Omega}\right) & 0 & 0 \\ -\frac{a\sqrt{\Phi}\Omega}{\rho^2} & 0 & 0 & 0 \end{vmatrix},$$

$$\gamma_{ab3} = \begin{vmatrix} 0 & 0 & -\frac{a\sqrt{\Phi}\Omega}{\rho^2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{\Phi}}{2}\left(\frac{2r}{\rho^2} - \frac{\Omega'}{\Omega}\right) \\ \frac{a\sqrt{\Phi}\Omega}{\rho^2} & 0 & 0 & \frac{\sqrt{\Omega}}{\rho\tan\theta} \\ 0 & -\frac{\sqrt{\Phi}}{2}\left(\frac{2r}{\rho^2} - \frac{\Omega'}{\Omega}\right) & -\frac{\sqrt{\Omega}}{\rho\tan\theta} & 0 \end{vmatrix}.$$

Taking in mind the above relations, we can specify the Dirac covariant equation (M designates the mass of the particle)

$$\left[i\gamma^a \left(e^\alpha_{(a)} \frac{\partial}{\partial x^\alpha} + \frac{1}{2}\sigma^{mn}\gamma_{mna} \right) - M \right] \Psi = 0, \tag{6}$$

in the generalized NUT space it takes the form:

$$\begin{gathered} \left[i\Big(\gamma^0 \frac{\rho}{\sqrt{\Delta}} + \gamma^3 \frac{2a}{\rho}\sqrt{\Omega}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\Big)\frac{\partial}{\partial t} \right. \\ -i\gamma^1\Big(\frac{\sqrt{\Delta}}{\rho}\frac{\partial}{\partial r} + \frac{r\sqrt{\Delta}}{2\rho^3} + \frac{\Delta'}{4\rho\sqrt{\Delta}} - \frac{\sqrt{\Delta}\Omega'}{2\rho\Omega}\Big) + i\gamma^0\gamma^2\gamma^3\frac{a\sqrt{\Delta}\Omega}{2\rho^3} \\ \left. -i\gamma^2\frac{\sqrt{\omega}}{\rho}\Big(\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta}\Big) - i\gamma^3\frac{\sqrt{\Omega}}{\rho}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - M \right] \Psi = 0. \end{gathered} \tag{7}$$

We assume the following form of the Dirac matrices

$$\gamma^0 = \begin{vmatrix} 0 & I \\ I & 0 \end{vmatrix}, \quad \gamma^1 = \begin{vmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{vmatrix}, \quad \gamma^2 = \begin{vmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{vmatrix}, \quad \gamma^3 = \begin{vmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{vmatrix},$$

$$i\gamma^0\gamma^2\gamma^3 = \begin{vmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{vmatrix}, \quad \Psi = \begin{vmatrix} \xi \\ \chi \end{vmatrix};$$

the Pauli matrices are designated as σ_i; the bispinor wave function consists of two spinor components

$$\sigma_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \sigma_2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \sigma_3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \quad \Psi = \begin{vmatrix} \xi \\ \chi \end{vmatrix}.$$

From (7), we derive equations in 2-spinor form

$$\begin{gathered} \sigma_1 \left(\frac{\sqrt{\Omega}}{\rho}\chi_{,2} + \frac{\sqrt{\Omega}}{2\rho\tan\theta}\chi \right) + \sigma_2 \left(\frac{\sqrt{\Omega}}{\rho\sin\theta}\chi_{,3} - \frac{2a\sqrt{\Omega}}{\rho}\tan\frac{\theta}{2}\,\chi_{,0} \right) \\ +\sigma_3 \left[\frac{\sqrt{\Delta}}{\rho}\chi_{,1} + \left(\frac{\Delta'}{4\rho\sqrt{\Delta}} + \frac{\sqrt{\Delta}}{2\rho^3}(x - ia\Omega) - \frac{\sqrt{\Delta}\Omega'}{2\Omega\rho} \right)\chi \right] + \frac{\rho}{\sqrt{\Delta}}\chi_{,0} + iM\xi = 0, \end{gathered} \tag{8}$$

$$\sigma_1\left(\frac{\sqrt{\Omega}}{\rho}\xi_{,2}+\frac{\sqrt{\Omega}}{2\rho\tan\theta}\xi\right)+\sigma_2\left(\frac{\sqrt{\Omega}}{\rho\sin\theta}\xi_{,3}-\frac{2a\sqrt{\Omega}}{\rho}\tan\frac{\theta}{2}\,\xi_{,0}\right)$$
$$+\sigma_3\left[\frac{\sqrt{\Delta}}{\rho}\xi_{,1}+\left(\frac{\Delta'}{4\rho\sqrt{\Delta}}+\frac{\sqrt{\Delta}}{2\rho^3}(x+ia\Omega)-\frac{\sqrt{\Delta}\Omega'}{2\Omega\rho}\right)\xi\right]-\frac{\rho}{\sqrt{\Delta}}\xi_{,0}-iM\chi=0; \tag{9}$$

where we apply the notations $\partial_\alpha =,\alpha$. Searching two spinors in the form

$$\xi=\Delta^{-1/4}\rho_+^{-1/2}e^{-i\epsilon t}e^{im\phi}X(r,\theta),\quad \chi=\Delta^{-1/4}\rho_-^{-1/2}e^{-i\epsilon t}e^{im\phi}Y(r,\theta), \tag{10}$$

with $\rho_+=r+ia,\quad \rho_-=r-ia$, we get

$$\sigma_1\sqrt{\Omega}D_\theta Y+i\sigma_2\sqrt{\Omega}HY+\sigma_3 D_{r-}Y-\frac{i\epsilon\rho^2}{\sqrt{\Delta}}Y+iM\rho_-X=0,$$
$$\sigma_1\sqrt{\Omega}D_\theta X+i\sigma_2\sqrt{\Omega}HX+\sigma_3 D_{r+}X+\frac{i\epsilon\rho^2}{\sqrt{\Delta}}X-iM\rho_+Y=0, \tag{11}$$

where

$$D_{r\pm}=\sqrt{\Delta}\frac{d}{dr}\pm\frac{i\sqrt{\Delta}}{2}\Big(\frac{a(1+\Omega)}{\rho^2}\pm\frac{i\Omega'}{\Omega}\Big),\quad D_\theta=\frac{\partial}{\partial\theta}+\frac{1}{2\tan\theta},\quad H=\frac{m}{\sin\theta}+2a\epsilon\tan\frac{\theta}{2}.$$

In the above equations, the variables can be separated within the substitutions

$$X_1=R_1(r)T_1(\theta),\quad X_2=R_2(r)T_2(\theta),\quad Y_1=R_3(r)T_1(\theta),\quad Y_2=R_4(r)T_2(\theta);$$

then we get the radial system

$$\Big(D_{r-}-\frac{i\epsilon\rho^2}{\sqrt{\Delta}}\Big)R_3+iM(r-ia)R_1=\Lambda\sqrt{\Omega}R_4,$$
$$\Big(D_{r-}+\frac{i\epsilon\rho^2}{\sqrt{\Delta}}\Big)R_4-iM(r-ia)R_2=\Lambda\sqrt{\Omega}R_3,$$
$$\Big(D_{r+}+\frac{i\epsilon\rho^2}{\sqrt{\Delta}}\Big)R_1-iM(r+ia)R_3=\Lambda\sqrt{\Omega}R_2,$$
$$\Big(D_{r+}-\frac{i\epsilon\rho^2}{\sqrt{\Delta}}\Big)R_2+iM(r+ia)R_4=\Lambda\sqrt{\Omega}R_1; \tag{12}$$

and the angular equations

$$\frac{dT_1}{d\theta}+\Big(\frac{1}{2\tan\theta}-\frac{m}{\sin\theta}-2a\epsilon\tan\frac{\theta}{2}\Big)T_1-\Lambda T_2=0,$$
$$\frac{dT_2}{d\theta}+\Big(\frac{1}{2\tan\theta}+\frac{m}{\sin\theta}+2a\epsilon\tan\frac{\theta}{2}\Big)T_2+\Lambda T_1=0, \tag{13}$$

where Λ is a separation constant.

The angular equation system is the same as for ordinary NUT space and has been solved previously in [6, 10]. Because of that, in the following we will study the radial equations only.

2. Radial equations, the massless case

In massless case, equations (12) simplify, being divided in two unlinked subsystems

$$\Big(\sqrt{\Delta}\frac{d}{dr}-\frac{i\sqrt{\Delta}}{2}\Big(\frac{a(1+\Omega)}{\rho^2}-\frac{i\Omega'}{\Omega}\Big)-\frac{i\epsilon\rho^2}{\sqrt{\Delta}}\Big)R_3=\Lambda\sqrt{\Omega}R_4, \tag{14}$$

$$\Big(\sqrt{\Delta}\frac{d}{dr}-\frac{i\sqrt{\Delta}}{2}\Big(\frac{a(1+\Omega)}{\rho^2}-\frac{i\Omega'}{\Omega}\Big)+\frac{i\epsilon\rho^2}{\sqrt{\Delta}}\Big)R_4=\Lambda\sqrt{\Omega}R_3, \tag{15}$$

$$\Big(\sqrt{\Delta}\frac{d}{dr}+\frac{i\sqrt{\Delta}}{2}\Big(\frac{a(1+\Omega)}{\rho^2}+\frac{i\Omega'}{\Omega}\Big)+\frac{i\epsilon\rho^2}{\sqrt{\Delta}}\Big)R_1=\Lambda\sqrt{\Omega}R_2, \tag{16}$$

$$\Big(\sqrt{\Delta}\frac{d}{dr}+\frac{i\sqrt{\Delta}}{2}\Big(\frac{a(1+\Omega)}{\rho^2}+\frac{i\Omega'}{\Omega}\Big)-\frac{i\epsilon\rho^2}{\sqrt{\Delta}}\Big)R_2=\Lambda\sqrt{\Omega}R_1. \tag{17}$$

We readily find the 2nd order equations for separate functions (further we utilize the dimensionless quantities, $x = \epsilon r, a \equiv \epsilon a$)

$$\begin{aligned}\Delta R_1'' + \frac{1}{2}\Big[\Delta\Big(\frac{2ia(1+\Omega)}{a^2+x^2} - \frac{3\Omega'}{\Omega}\Big) + \Delta'\Big]R_1' + \Big[-\Lambda^2\Omega + 2ix + \frac{(a^2+x^2)^2}{\Delta} \\ -\frac{a^2\Delta(1+\Omega)^2}{4(a^2+x^2)^2} - \frac{iax\Delta(1+\Omega)}{(a^2+x^2)^2} + \frac{ia\Delta'(1+\Omega)}{4(a^2+x^2)} - \frac{i(a^2+x^2)\Delta'}{2\Delta} - \frac{ia\Delta(3+\Omega)\Omega'}{4\Omega(x^2+a^2)} \\ -\frac{i(x^2+a^2)\Omega'}{2\Omega} - \frac{\Delta'\Omega'}{4\Omega} + \frac{\Delta\Omega'^2}{\Omega^2} - \frac{\Delta\Omega''}{2\Omega}\Big]R_1 = 0,\end{aligned} \tag{18}$$

$$\begin{aligned}\Delta R_2'' + \frac{1}{2}\Big[\Delta\Big(\frac{2ia(1+\Omega)}{a^2+x^2} - \frac{3\Omega'}{\Omega}\Big) + \Delta'\Big]R_2' + \Big[-\Lambda^2\Omega - 2ix + \frac{(a^2+x^2)^2}{\Delta} \\ -\frac{a^2\Delta(1+\Omega)^2}{4(a^2+x^2)^2} - \frac{iax\Delta(1+\Omega)}{(a^2+x^2)^2} + \frac{ia\Delta'(1+\Omega)}{4(a^2+x^2)} + \frac{i(a^2+x^2)\Delta'}{2\Delta} - \frac{ia\Delta(3+\Omega)\Omega'}{4\Omega(x^2+a^2)} \\ +\frac{i(x^2+a^2)\Omega'}{2\Omega} - \frac{\Delta'\Omega'}{4\Omega} + \frac{\Delta\Omega'^2}{\Omega^2} - \frac{\Delta\Omega''}{2\Omega}\Big]R_2 = 0.\end{aligned} \tag{19}$$

We search solutions in the form

$$R_1 = Z_1/V, \qquad V = (x-ia)^{\alpha}(x+ia)^{\beta}(x-x_1)^{\gamma}(x-x_2)^{\sigma}e^{-ix\omega},$$

where x_1, x_2 are the roots of equation

$$\Delta = x^2 - 2Mx - a^2 = (x-x_1)(x-x_2) = 0. \tag{20}$$

The substitution gives the equation for Z_1 as follows

$$\begin{aligned}Z_1'' + \frac{1}{2}\Big(4i\omega + \frac{1-4\alpha}{x-ia} - \frac{1+4\beta}{x+ia} + \frac{1-4\gamma}{x-x_1} + \frac{1-4\sigma}{x-x_2} + \frac{2ia\Omega}{x^2+a^2} - \frac{3\Omega'}{\Omega}\Big)Z_1' \\ +\Big(A + \frac{B}{x-ia} + \frac{C}{x+ia} + \frac{D}{x-x_1} + \frac{E}{x-x_2} \\ +\frac{F}{(x-ia)^2} + \frac{G}{(x+ia)^2} + \frac{H}{(x-x_1)^2} + \frac{K}{(x-x_2)^2}\Big)Z_1 = 0.\end{aligned} \tag{21}$$

Here the explicit form of coefficients $A, B, C, D, E, F, G, H, K$ are determined by the choose of the function Ω and can be expressed from equation (18). We will search such functions Ω which give the solution in Heun functions. So, they should be regular at singular points of black hole horizon x_1, x_2. Because of that, we search the function Ω in the form

$$\Omega = (x-ia)^z(x+ia)^v(x-x_1)^q(x-x_2)^s.$$

With this substitution the coefficient at Z_1' takes the form

$$\begin{aligned}2i\omega + \frac{1-4\alpha-3z}{2(x-ia)} - \frac{1+4\beta+3v}{2(x+ia)} + \frac{1-4\gamma-3q}{2(x-x_1)} + \frac{1-4\sigma-3s}{2(x-x_2)} \\ +\frac{ia(x-ia)^z(x+ia)^v(x-x_1)^q(x-x_2)^s}{x^2+a^2}.\end{aligned}$$

We need to take the exponents z, v, q, s and $\alpha, \beta, \gamma, \sigma$ in such a way to terms with the multipliers $(x \pm ia)^{-1}$ vanish, as well as B, C, F, G, H, K in equation (21) also should vanish.

At $z = v = q = s = 0$ the metric reduces to original NUT solution. Applying $\alpha = 1/2$, $\beta = -1/2$, $\omega = \pm 1$, γ and σ are determined by equations:

$$2\gamma^2 + \gamma - ix_1 + 2x_1^2 = 0 \quad \Longrightarrow \quad \gamma_1 = ix_1,\ \gamma_2 = -1/2 - ix_1,$$

$$2\sigma^2 + \sigma - ix_2 + 2x_2^2 = 0 \quad \Longrightarrow \quad \sigma_1 = ix_2,\ \sigma_2 = -1/2 - ix_2;$$

we get the equation (21) in the form

$$Z_1'' + \frac{1}{2}\Big(4i\omega + \frac{1-4\gamma}{x-x_1} + \frac{1-4\sigma}{x-x_2}\Big)Z_1' + \tag{22}$$
$$\frac{-2\Lambda^2 + 4a^2 + 4\gamma\sigma - (\gamma+\sigma) - 2i\omega M + 4i\omega\,(\gamma x_2 + \sigma x_1) + 2x\,(-2i\omega(\gamma+\sigma) + 4M + i(\omega+1))}{2\,(r-x_1)\,(r-x_2)} Z_1 = 0;$$

its general solution can be expressed in terms of confluent Heun functions [6].

The choice for $v \neq 0$, $z \neq 0$ does not allow to remove coefficients B, C, F, G in equation (21). Appropriate choices, which allow to obtain the solution in the confluent Heun functions were found at $v = z = 0$, $q = s = 1$ ($\Omega = \Delta$) and $q = s = -1$ ($\Omega = \Delta^{-1}$). However, only the first case gives appropriate solution at all separation parameter Λ. This choice gives the equation (21) in the form

$$Z_1'' + \Big(i(2\omega + a) - \frac{1+2\gamma}{x-x_1} - \frac{1+2\sigma}{x-x_2}\Big)Z_1'$$
$$+\frac{1}{2(x-x_1)(x-x_2)}\Big(4a^2 + (1+2\sigma)(1+2\gamma+i(2\omega+a)x_1) + i(1+2\gamma)(2\omega+a)x_2$$
$$+2x(4M - i(1+\gamma+\sigma)(2\omega+a))\Big)Z_1 = 0,$$
$$\alpha = \frac{1}{4}\Big(1 - 2a^2 - 2iaM\Big),\ \beta = -\frac{1}{4}\Big(1 - 2a^2 + 2iaM\Big),\ \omega = \pm\sqrt{1-\Lambda^2} - \frac{a}{2},$$
$$\gamma_1 = \frac{-1+2ix_1}{2},\ \gamma_2 = \frac{-3-2ix_1}{2},\quad \sigma_1 = \frac{-1+2ix_2}{2},\ \sigma_2 = \frac{-3-2ix_2}{2}.$$

Applying the variable

$$y = \frac{x-x_2}{x_1-x_2},$$

we get the equation of confluent Heun type:

$$Z_1'' + \Big(A - \frac{1+2\gamma}{y-1} - \frac{1+2\sigma}{y}\Big)Z_1' + \frac{B+Cy}{2y(y-1)}Z_1 = 0,$$

where

$$A = i\,(x_1-x_2)\,(a+2\omega),\quad C = -2i\,(x_1-x_2)\,(a(\gamma+\sigma+1) + 2\omega(\gamma+\sigma+1) + 4iM)\,,$$
$$B = -2a^2 + (2\sigma+1)\,(ia\,(x_1-x_2) + 2\gamma + 2ix_1\omega + 1) - ix_2\omega(2\gamma+4\sigma+3) + 4x_2^2.$$

Solution of this equation is

$$Z_1 = c_1 \mathrm{HeunC}\left[-\frac{B}{2}, \frac{C}{2}, -(2\sigma+1), -(2\gamma+1), A, y\right]$$
$$+c_2 y^{2(\sigma+1)} \mathrm{HeunC}\left[(2\sigma+2)(A+2\gamma+1) - \frac{B}{2}, 2A(\sigma+1) + \frac{C}{2}, 2\sigma+3, -(2\gamma+1), A, y\right].$$

Conclusion

We consider a set of the metrics derived as a generalization of NUT spacetime. We have derived the Dirac equation on the background of the generalized metric with NUT parameter, have performed the variable separation, and have found the systems for angular and radial components. We demonstrate that the angular equations preserve the form of the equations for the original NUT metric, this fact evidences the preserving of angular symmetry of the original NUT metric.

We have studied the radial equations in the case of massless particle and have found the special choice of the generalized NUT metric in such a way to solve the radial equations in terms of confluent Heun functions.

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